Forecasts Based on Conditional Expectation

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1 Questions

1) Which series can be forecast?
2) How to forecast time series?
3) Unconditional moments vs. Conditional moments
4) Variance vs. Forcast error variance
5) State varibles vs. Control variables (rf. McCandless 2008, p.51)

2 MSE

A quadratic loss function is choosed to

$$min\, \mathrm{MSE}(y^*_{t+1|t}) \equiv \mathbb{E}(y_t - y^*_{t+1|t})^2, \quad \text{where } y^*_{t+1|t} = \mathbb{E}(y_{t+1}|y_t, y_{t-1}, \ldots) \equiv \mathbb{E}_t y_{t+1}.$$

The MSE of this optimal forecast is

$$\mathbb{E}(y_{t+1} - \mathbb{E}_t y_{t+1})^2.$$

See Hamilon (1994, pp.72-73)

3 Forecasts of ARMA Models

1) MA(∞)

$$y_t = \sum_{i=0}^{\infty} \theta_i \epsilon_{t-i} = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots, \quad \theta_0 \equiv 1.$$

$$\xrightarrow[]{\text{move fowward}}\begin{cases} y_{t+1} = \sum_{i=0}^{\infty}\theta_i\epsilon_{t+1-i} = \epsilon_{t+1} + \theta_1\epsilon_t + \theta_2\epsilon_{t-1} + \theta_3\epsilon_{t-2} + \cdots, \\ y_{t+2} = \sum_{i=0}^{\infty}\theta_i\epsilon_{t+2-i} = \epsilon_{t+2} + \theta_1\epsilon_{t+1} + \theta_2\epsilon_t + \theta_3\epsilon_{t-1} + \cdots, \\ \vdots \\ y_{t+h} = \sum_{i=0}^{\infty}\theta_i\epsilon_{t+h-i} = \epsilon_{t+h} + \theta_1\epsilon_{t+h-1} + \theta_2\epsilon_{t+h-2} + \cdots + \theta_h\epsilon_t + \cdots + \theta_{h+1}\epsilon_{t-1} + \cdots; \end{cases}$$

$$\xrightarrow[]{\text{forecast ahead}}\begin{cases} \mathbb{E}y_t = 0, \\ \mathbb{E}_t y_t = \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} + \cdots, \\ \mathbb{E}_t y_{t+1} = \mathbb{E}_t(\epsilon_{t+1} + \theta_1\epsilon_t + \theta_2\epsilon_{t-1} + \theta_3\epsilon_{t-2} + \cdots) = 0 + \theta_1\epsilon_t + \theta_2\epsilon_{t-1} + \theta_3\epsilon_{t-2} + \cdots, \\ \mathbb{E}_t y_{t+2} = \mathbb{E}_t(\epsilon_{t+2} + \theta_1\epsilon_{t+1} + \theta_2\epsilon_t + \theta_3\epsilon_{t-1} + \cdots) = 0 + 0 + \theta_2\epsilon_t + \theta_3\epsilon_{t-1} + \cdots, \\ \vdots \\ \mathbb{E}_t y_{t+h} = \mathbb{E}_t(\epsilon_{t+h} + \theta_1\epsilon_{t+h-1} + \cdots + \theta_h\epsilon_t + \theta_{h+1}\epsilon_{t-1} + \cdots) = \theta_h\epsilon_t + \theta_{h+1}\epsilon_{t-1} + \cdots; \end{cases}$$

$$\xrightarrow[]{\text{variance}}\begin{cases} \text{var}y_t = \mathbb{E}[(y_t - \mathbb{E}y_t)^2] = \mathbb{E}(y_t^2) = (1 + \theta_1^2 + \theta_2^2 + \cdots)\sigma_\epsilon^2, \\ \text{var}_t y_t = \mathbb{E}_t[(y_t - \mathbb{E}_t y_t)^2] = 0, \\ \text{var}_t y_{t+1} = \mathbb{E}_t[(y_{t+1} - \mathbb{E}_t y_{t+1})^2] = \mathbb{E}_t\epsilon_{t+1}^2 = \sigma_\epsilon^2, \\ \text{var}_t y_{t+2} = \mathbb{E}_t[(y_{t+2} - \mathbb{E}_t y_{t+2})^2] = \mathbb{E}_t(\epsilon_{t+2} + \theta_1\epsilon_{t+1})^2 = (1 + \theta_1^2)\sigma_\epsilon^2, \\ \vdots \\ \text{var}_t y_{t+h} = \mathbb{E}_t[(y_{t+h} - \mathbb{E}_t y_{t+h})^2] = (1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_{h-1}^2)\sigma_\epsilon^2. \end{cases}$$

$$\begin{cases} \mathbb{E}_t y_{t+h} \equiv \mathbb{E}(y_{t+h}|y_t, y_{t-1}, y_{t-2}, \ldots, \epsilon_t, \epsilon_{t-1}, \epsilon_{t-2}, \ldots) \leftarrow \text{form predictions of } y \text{ given its past} \\ \text{var}_t y_{t+h} \equiv \text{var}(y_{t+h}|y_t, y_{t-1}, y_{t-2}, \ldots, \epsilon_t, \epsilon_{t-1}, \epsilon_{t-2}, \ldots) \leftarrow \text{how certain about the prediction} \end{cases}$$

2) AR(1)

$$y_t = \phi y_{t-1} + \epsilon_t = \sum_{i=0}^{\infty}\phi^i\epsilon_{t-i}.$$

$$\xRightarrow{\text{move fowward}} \begin{cases} y_{t+1} = \phi y_t + \epsilon_{t+1}, \\ y_{t+2} = \phi y_{t+1} + \epsilon_{t+2} = \phi(\phi y_t + \epsilon_{t+1}) + \epsilon_{t+2} = \phi^2 y_t + \phi\epsilon_{t+1} + \epsilon_{t+2}, \\ \vdots \\ y_{t+h} = \phi y_{t+h-1} + \epsilon_{t+h}; \end{cases}$$

$$\xrightarrow{\text{unconditional expectation}} \mathbb{E} y_t = 0,$$

$$\xrightarrow{\text{conditional expectation}} \mathbb{E}_t y_t = \phi y_{t-1},$$

$$\xRightarrow{\text{1 step ahead forecast}} \mathbb{E}_t y_{t+1} = \mathbb{E}_t(\phi y_t) + \mathbb{E}_t \epsilon_{t+1} = \phi \mathbb{E}_t y_t + 0 = \phi y_t,$$

$$\xRightarrow{\text{2 steps ahead forecasts}} \mathbb{E}_t y_{t+2} = \mathbb{E}_t(\phi y_{t+1}) = \phi \mathbb{E}_t y_{t+1} = \phi(\phi y_t) = \phi^2 y_t,$$

$$\vdots$$

$$\xRightarrow{\text{h steps ahead forecasts}} \mathbb{E}_t y_{t+h} = \mathbb{E}_t(\phi y_{t+h-1}) = \phi \mathbb{E}_t y_{t+h-1} = \phi^h y_t;$$

$$\xRightarrow{\text{forecast error variances}} \begin{cases} \text{var} y_t = \mathbb{E}[(y_t - \mathbb{E} y_t)^2] = \mathbb{E} y_t^2 = \mathbb{E}[(\epsilon_t + \phi\epsilon_{t-1} + \phi^2\epsilon_{t-2} + \cdots)^2] = \frac{\sigma_\epsilon^2}{1-\phi^2}, \\ \text{var}_t y_t = \mathbb{E}_t[(y_t - \mathbb{E}_t y_t)^2] = \mathbb{E}_t[(y_t - \phi y_{t-1})^2] = \mathbb{E}_t \epsilon_t^2 = \sigma_\epsilon^2, \\ \text{var}_t y_{t+1} = \mathbb{E}_t[(y_{t+1} - \mathbb{E}_t y_{t+1})^2] = \mathbb{E}_t[(y_{t+1} - \phi y_t)^2] = \sigma_\epsilon^2, \\ \text{var}_t y_{t+2} = \mathbb{E}_t[(y_{t+2} - \phi^2 y_t)^2] = \mathbb{E}_t[(\phi\epsilon_{t+1} + \epsilon_{t+2})^2] = (1+\phi^2)\sigma_\epsilon^2, \\ \vdots \\ \text{var}_t y_{t+h} = \mathbb{E}_t[(y_{t+h} - \phi^h y_t)^2] = (1 + \phi^2 + \phi^4 + \cdots + \phi^{2(h-1)})\sigma_\epsilon^2. \end{cases}$$

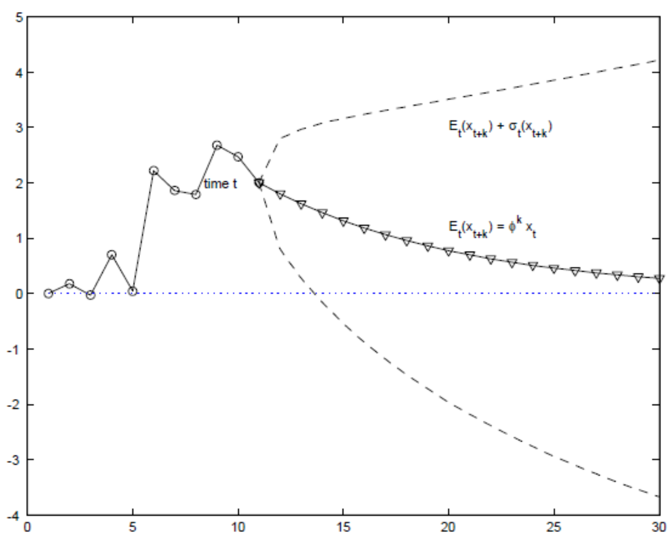

Figure 1: AR(1) forecast and standard deviation [1]

4 Forecasts of VAR Processes

1) Vector MA(∞)

$$\mathbf{y}_t = \sum_{i=0}^{\infty} \boldsymbol{\theta}_i \boldsymbol{\epsilon}_{t-i} = \boldsymbol{\epsilon}_t + \boldsymbol{\theta}_1 \boldsymbol{\epsilon}_{t-1} + \boldsymbol{\theta}_2 \boldsymbol{\epsilon}_{t-2} + \cdots, \quad \boldsymbol{\theta}_0 \equiv \mathbf{I}.$$

[1] Source: Cochrane (2005, p.33)

$$\mathbb{E}_t \mathbf{y}_{t+h} = \boldsymbol{\theta}_h \boldsymbol{\epsilon}_t + \boldsymbol{\theta}_{h+1} \boldsymbol{\epsilon}_{t-1} + \boldsymbol{\theta}_{h+2} \boldsymbol{\epsilon}_{t-2} + \cdots ;$$
$$\text{var}_t \mathbf{y}_{t+h} = \boldsymbol{\sigma}^2 + \boldsymbol{\theta}_1 \boldsymbol{\sigma}^2 \boldsymbol{\theta}_1' + \cdots + \boldsymbol{\theta}_{h-1} \boldsymbol{\sigma}^2 \boldsymbol{\theta}_{h-1}'.$$

2) VAR(1)

$$\mathbf{y}_t = \mathbf{F}\mathbf{y}_{t-1} + \mathbf{G}\boldsymbol{\epsilon}_t \quad \Leftrightarrow \quad \mathbf{y}_t = \mathbf{F}\mathbf{y}_{t-1} + \mathbf{G}\boldsymbol{\xi}_t.$$

$$\mathbf{G} \equiv \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} \sigma^2 \\ 0 \\ \sigma^2 \end{bmatrix} \qquad \mathbb{E}(\boldsymbol{\epsilon}_t \boldsymbol{\epsilon}_t') = \boldsymbol{\sigma}^2 \to \mathbb{E}\boldsymbol{\xi}_t \boldsymbol{\xi}_t' = \mathbf{I}.$$

$$\mathbf{y}_t = \mathbf{F}\mathbf{y}_{t-1} + \mathbf{G}\boldsymbol{\xi}_t = \sum_{i=0}^{\infty} \mathbf{F}^i \mathbf{G} \boldsymbol{\xi}_{t-i}.$$

$$\begin{aligned}
\mathbf{y}_{t+1} &= \mathbf{F}\mathbf{y}_t + \mathbf{G}\boldsymbol{\xi}_{t+1} \\
\mathbf{y}_{t+2} &= \mathbf{F}\mathbf{y}_{t+1} + \mathbf{G}\boldsymbol{\xi}_{t+2} = \mathbf{F}^2 \mathbf{y}_t + \mathbf{F}\mathbf{G}\boldsymbol{\xi}_{t+1} + \mathbf{G}\boldsymbol{\xi}_{t+2} \\
&\vdots \\
\mathbf{y}_{t+h} &= \mathbf{F}\mathbf{y}_{t+h-1} + \mathbf{G}\boldsymbol{\xi}_{t+h}.
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}_t \mathbf{y}_{t+1} &= \mathbb{E}_t(\mathbf{F}\mathbf{y}_t) = \mathbf{F}\mathbb{E}_t \mathbf{y}_t = \mathbf{F}\mathbf{y}_t; \\
\mathbb{E}_t \mathbf{y}_{t+2} &= \mathbb{E}_t(\mathbf{F}\mathbf{y}_{t+1}) = \mathbf{F}\mathbb{E}_t \mathbf{y}_{t+1} = \mathbf{F}^2 \mathbf{y}_t; \\
&\vdots \\
\mathbb{E}_t \mathbf{y}_{t+h} &= \mathbb{E}_t(\mathbf{F}\mathbf{y}_{t+h-1}) = \mathbf{F}\mathbb{E}_t \mathbf{y}_{t+h-1} = \mathbf{F}^h \mathbf{y}_t.
\end{aligned}$$

$$\begin{aligned}
\text{var}_t \mathbf{y}_{t+1} &= \mathbb{E}_t[(\mathbf{y}_{t+1} - \mathbb{E}_t \mathbf{y}_{t+1})(\mathbf{y}_{t+1} - \mathbb{E}_t \mathbf{y}_{t+1})'] = \mathbb{E}_t[(\mathbf{G}\boldsymbol{\epsilon}_{t+1})(\mathbf{G}\boldsymbol{\epsilon}_{t+1})') = \mathbf{G}\mathbf{G}', \\
\text{var}_t \mathbf{y}_{t+2} &= \mathbb{E}_t[(\mathbf{y}_{t+2} - \mathbb{E}_t \mathbf{y}_{t+2})(\mathbf{y}_{t+2} - \mathbb{E}_t \mathbf{y}_{t+2})'] = \mathbb{E}_t[(\mathbf{F}\mathbf{G}\boldsymbol{\epsilon}_{t+1} + \mathbf{G}\boldsymbol{\epsilon}_{t+2})(\mathbf{F}\mathbf{G}\boldsymbol{\epsilon}_{t+1} + \mathbf{G}\boldsymbol{\epsilon}_{t+2})'] = \mathbf{F}\mathbf{G}\mathbf{G}'\mathbf{F} + \mathbf{G}\mathbf{G}', \\
&\vdots \\
\text{var}_t \mathbf{y}_{t+h} &= \mathbb{E}_t[(\mathbf{y}_{t+h} - \mathbb{E}_t \mathbf{y}_{t+h})(\mathbf{y}_{t+h} - \mathbb{E}_t \mathbf{y}_{t+h})'] = \sum_{i=0}^{h-1} \mathbf{F}^h \mathbf{G}\mathbf{G}' \mathbf{F}^{h'}.
\end{aligned}$$

$$\xrightarrow{\text{programming in a simple loop}} \begin{cases} \mathbb{E}_t \mathbf{y}_{t+h} = \mathbf{F}\mathbb{E}_t \mathbf{y}_{t+h-1}, \\ \text{var}\mathbf{y}_{t+h} = \mathbf{F}\text{var}\mathbf{y}_{t+h-1}\mathbf{F}' + \mathbf{G}\mathbf{G}'. \end{cases}$$

5 The State-Space Representation

Any process can be mapped into a VAR(1), which leads to easy programming of forecasts.

1) MA(1)→VAR(1)

$$y_t = c + \epsilon_t + \theta \epsilon_{t-1} \xLeftrightarrow{\mu \equiv \mathbb{E}y_t = c} y_t = \mu + \epsilon_t + \theta \epsilon_{t-1}.$$

$$\begin{aligned}
\begin{bmatrix} \epsilon_{t+1} \\ \epsilon_t \end{bmatrix} &\overset{\text{state equation}}{=} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \epsilon_t \\ \epsilon_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{t+1} \\ 0 \end{bmatrix}, \\
y_t &\overset{\text{observation eq.}}{=} \mu + [1 \quad \theta] \begin{bmatrix} \epsilon_t \\ \epsilon_{t-1} \end{bmatrix}.
\end{aligned}$$

2) AR(p)→VAR(1)

$$\begin{cases} y_t - \mu = \phi_1(y_{t-1} - \mu) + \phi_2(y_{t-2} - \mu) + \cdots + \phi_p(y_{t-p} - \mu) + \epsilon_t, \\ y_{t+1} - \mu = \phi_1(y_t - \mu) + \phi_2(y_{t-1} - \mu) + \cdots + \phi_p(y_{t+1-p} - \mu) + \epsilon_{t+1}. \end{cases}$$

$$\begin{bmatrix} y_{t+1} - \mu \\ y_t - \mu \\ \vdots \\ y_{t+1-(p-2)} - \mu \\ y_{t+1-(p-1)} - \mu \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} y_t - \mu \\ y_{t-1} - \mu \\ \vdots \\ y_{t+1-(p-1)} - \mu \\ y_{t+1-p} - \mu \end{bmatrix} + \begin{bmatrix} \epsilon_{t+1} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$y_t = \quad \mu \quad + \quad [1 \quad 0 \quad \cdots \quad 0] \begin{bmatrix} y_t - \mu \\ y_{t-1} - \mu \\ \vdots \\ y_{t+1-(p-1)} - \mu \\ y_{t+1-p} - \mu \end{bmatrix}$$

3) VAR(∞)→VAR(1)

$$\begin{cases} y_t = \phi_{yy}^{(1)} y_{t-1} + \phi_{yy}^{(2)} y_{t-2} + \cdot + \phi_{yz}^{(1)} z_{t-1} + \phi_{yz}^{(2)} z_{t-2} + \cdots + \epsilon_{yt}, \\ z_t = \phi_{zy}^{(1)} y_{t-1} + \phi_{zy}^{(2)} y_{t-2} + \cdot + \phi_{zz}^{(1)} z_{t-1} + \phi_{zz}^{(2)} z_{t-2} + \cdots + \epsilon_{zt} \end{cases}$$

$$\begin{bmatrix} y_t \\ z_t \\ y_{t-1} \\ z_{t-1} \\ \vdots \end{bmatrix} = \begin{bmatrix} \phi_{yy}^{(1)} & \phi_{yz}^{(1)} & \phi_{yy}^{(2)} & \phi_{yz}^{(2)} & \cdots \\ \phi_{zy}^{(1)} & \phi_{zz}^{(1)} & \phi_{zy}^{(2)} & \phi_{zz}^{(2)} & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} y_{t-1} \\ z_{t-1} \\ y_{t-2} \\ z_{t-2} \\ \vdots \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} \epsilon_{yt} \\ \epsilon_{zt} \end{bmatrix}$$

$$\mathbf{x}_t = \quad \mathbf{F} \quad \mathbf{x}_{t-1} \quad + \quad \mathbf{G}\boldsymbol{\epsilon}_t \begin{cases} \mathbb{E}\boldsymbol{\epsilon}_t = \mathbf{0} \\ \mathbb{E}(\boldsymbol{\epsilon}_t \boldsymbol{\epsilon}_t') = \boldsymbol{\sigma}^2 \end{cases}$$

4) ARMA(2, 1)→VAR(1)

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t + \theta_1 \epsilon_{t-1}.$$

$$\begin{bmatrix} y_t \\ y_{t-1} \\ \epsilon_t \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & \theta_1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ \epsilon_{t-1} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \epsilon_t.$$

$$\mathbf{y}_t = \mathbf{F}\mathbf{y}_{t-1} + \mathbf{G}\epsilon_t \quad \Leftrightarrow \quad \mathbf{y}_t = \mathbf{F}\mathbf{y}_{t-1} + \mathbf{G}\xi_t.$$

$$\mathbf{G} \equiv \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} \sigma_\epsilon^2 \\ 0 \\ \sigma_\epsilon^2 \end{bmatrix} \qquad \mathbb{E}\epsilon_t^2 = \sigma_\epsilon^2 \to \mathbb{E}\xi_t^2 = 1.$$

5) ARMA(2, 2)→VAR(1)

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}.$$

$$\begin{bmatrix} y_t \\ y_{t-1} \\ \epsilon_t \\ \epsilon_{t-1} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & \theta_1 & \theta_2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ \epsilon_{t-1} \\ \epsilon_{t-2} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \epsilon_t.$$

6) ARMA(p, q)→VAR(1)

$$\begin{cases} y_t - \mu = \phi_1(y_{t-1} - \mu) + \phi_2(y_{t-2} - \mu) + \cdots + \phi_p(y_{t-p} - \mu) + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} + \cdots + \theta_q\epsilon_{t-q}, \\ y_{t+1} - \mu = \phi_1(y_t - \mu) + \phi_2(y_{t-1} - \mu) + \cdots + \phi_p(y_{t+1-p} - \mu) + \epsilon_{t+1} + \theta_1\epsilon_t + \theta_2\epsilon_{t-1} + \cdots + \theta_q\epsilon_{t+1-q}. \end{cases}$$

$$\begin{bmatrix} y_{t+1} - \mu \\ y_t - \mu \\ \vdots \\ y_{t+1-(p-2)} - \mu \\ y_{t+1-(p-1)} - \mu \\ \epsilon_{t+1} \\ \epsilon_t \\ \vdots \\ \epsilon_{t+1-(q-2)} \\ \epsilon_{t+1-(q-1)} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p & \theta_1 & \theta_2 & \cdots & \theta_{q-1} & \theta_q \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} y_t - \mu \\ y_{t-1} - \mu \\ \vdots \\ y_{t+1-(p-1)} - \mu \\ y_{t+1-p} - \mu \\ \epsilon_t \\ \epsilon_{t-1} \\ \vdots \\ \epsilon_{t+1-(q-1)} \\ \epsilon_{t+1-q} \end{bmatrix} + \begin{bmatrix} \epsilon_{t+1} \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$y_t = \mu \quad + \quad [1 \quad 0 \quad \cdots \quad 0 \quad 1 \quad \theta_1 \quad \cdots \quad \theta_{q-1}] \begin{bmatrix} y_t - \mu \\ y_{t-1} - \mu \\ \vdots \\ y_{t+1-(p-1)} - \mu \\ y_{t+1-p} - \mu \\ \epsilon_t \\ \epsilon_{t-1} \\ \vdots \\ \epsilon_{t+1-(q-1)} \\ \epsilon_{t+1-q} \end{bmatrix}$$

6) ARMA(2, 2) with two variables y_t and z_t→VAR(1)

$$\begin{cases} y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}, \\ z_t = \rho z_{t-1} + \epsilon_t. \end{cases}$$

$$\begin{bmatrix} y_t \\ y_{t-1} \\ z_t \\ \epsilon_t \\ \epsilon_{t-1} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & 0 & \theta_1 & \theta_2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ z_{t-1} \\ \epsilon_{t-1} \\ \epsilon_{t-2} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \epsilon_t.$$

6 How to Calculate the Impulse-Response

The impulse response function is the path that y follows if it is kicked by a single unit shock. It allows us to start thinking about “causes” and “effects”.

1) MA(∞) or AR(1)

$$\begin{cases} y_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots = \sum_{i=0}^{\infty} \theta_i \epsilon_{t-i}, \quad \theta_0 \equiv 1; \\ y_t = \phi y_{t-1} + \epsilon_t \overset{\text{the closed form solution}}{=} \sum_{i=0}^{\infty} \phi^i \epsilon_{t-i}. \end{cases}$$

Table 1: IRF of two simple processes

MA(∞)	ϵ_t:	0	0	1	0	0	0	$\cdots$
	y_t :	0	0	1	θ_0	θ_1	θ_2	$\cdots$
AR(1)	ϵ_t:	0	0	1	0	0	0	$\cdots$
	y_t :	0	0	1	ϕ	ϕ^2	ϕ^3	$\cdots$

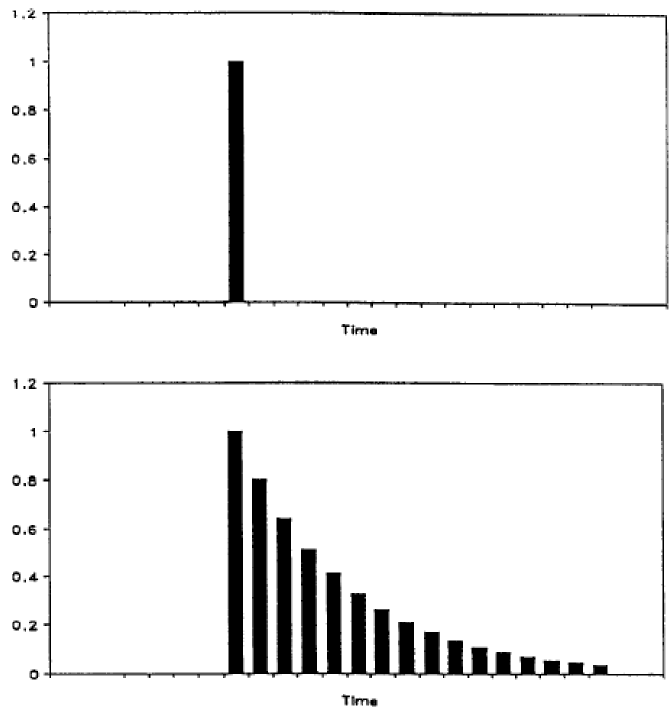

Figure 2: Value of y when it's kicked by a single unit shock ϵ_t (i.e., $\epsilon_{t-h} = \epsilon_{t+h} = 0$ but $\epsilon_t = 1$)[1]

2) Vector MA(∞) or VAR(1)

$$\mathbf{x}_t = (\mathbf{B}_0 + \mathbf{B}_1 L + \mathbf{B}_2 L^2 + \cdots)\boldsymbol{\epsilon}_t \equiv \mathbf{B}(L)\boldsymbol{\epsilon}_t \quad \Leftrightarrow \quad \begin{bmatrix} y_t \\ z_t \end{bmatrix} = \begin{bmatrix} \theta_{yy}(L) & \theta_{yz}(L) \\ \theta_{zy}(L) & \theta_{zz}(L) \end{bmatrix} \begin{bmatrix} \epsilon_{yt} \\ \epsilon_{zt} \end{bmatrix},$$

so $\theta_{yy}(L)$ gives the response of y_{t+h} to a unit shock ϵ_{yt}, and $\theta_{yz}(L)$ gives the response of y_{t+h} to a unit shock ϵ_{zt}.

$$\mathbf{x}_t = \mathbf{F}\mathbf{x}_{t-1} + \mathbf{G}\boldsymbol{\epsilon}_t = \sum_{i=0}^{\infty} \mathbf{F}^i \mathbf{G} \boldsymbol{\epsilon}_{t-i} \xrightarrow{\sum_{t=0}^{\infty} \beta^t \frac{\partial \mathbf{x}_t}{\partial \epsilon_0}} = \mathbf{G}, \mathbf{FG}, \mathbf{F}^2\mathbf{G}, \ldots, \mathbf{F}^i\mathbf{G}, \ldots \leftarrow \epsilon_0 = 1, \, \beta = 1.$$

7 Numerical Solution Using Dynare

see Miao(2014, pp.32,69)

[1]Source: Hamilton (1994, p.5)