# Time Series Analysis

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# Outline

# The modern use of TSMs: hypothesis testing

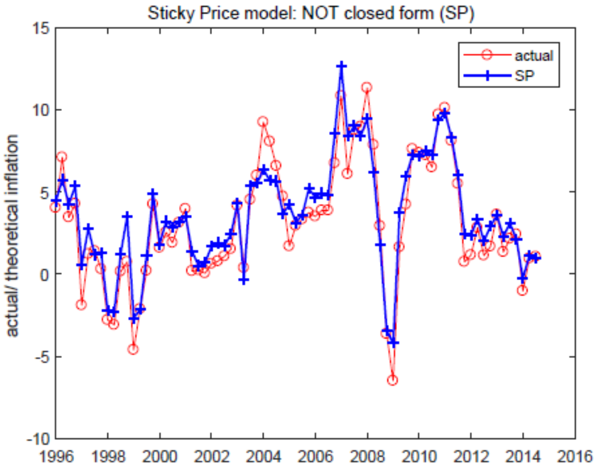


Figure: Real $\pi_t$ and predicted $\pi_t$ by sticky price theory

Source: Deng (2018, PhD thesis, ch.4.)

# Cont'd

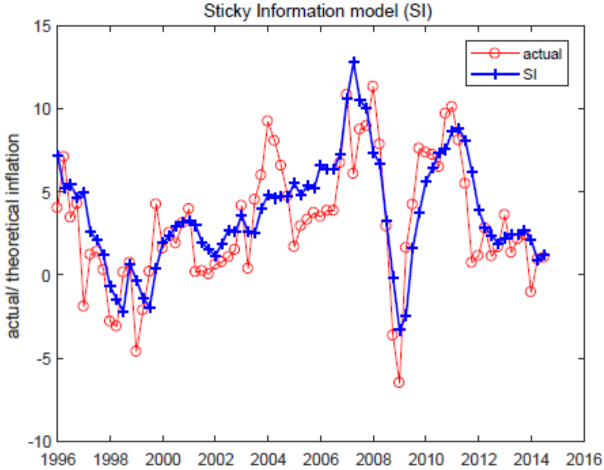


Figure: Real $\pi_t$ and predicted $\pi_t$ by sticky information theory

Source: Deng (2018, PhD thesis, ch.4.)

# Cont'd: theoretical hypotheses

$$
AS\begin{cases}
\text{sticky price} & \text{sticky information} \\
p_t^* = p_t + \alpha y_t, & p_t^* = p_t + \alpha y_t, \\
x_t^{k=0} = \lambda \sum_{j=0}^{\infty} (1-\lambda)^j \mathbb{E}_t p_{t+j}^* = \lambda p_t^* + (1-\lambda)\mathbb{E}_t x_{t+1}, & x_t^k = \mathbb{E}_{t-k} p_t^*,\ k = [0, \infty) \\
p_t = \lambda \sum_{j=0}^{\infty} (1-\lambda)^j x_{t-j} = \lambda x_t + (1-\lambda) p_{t-1}. & p_t = \lambda \sum_{k=0}^{\infty} (1-\lambda)^k x_t^k = \lambda \sum_{k=0}^{\infty} (1-\lambda)^k \mathbb{E}_{t-k}(p_t + \alpha y_t). \\
\Rightarrow \quad x_t = \frac{1}{\lambda} p_t - \frac{1-\lambda}{\lambda} p_{t-1} = \frac{1}{\lambda}\pi_t + p_{t-1}, & = \lambda(p_t + \alpha y_t) + \lambda \sum_{\mathbf{k=0}}^{\infty} (1-\lambda)^{k+\mathbf{1}} \mathbb{E}_{t-\mathbf{1}-k}(p_t + \alpha y_t), \\
\Rightarrow \quad \mathbb{E}_t x_{t+1} = \frac{1}{\lambda}\mathbb{E}_t \pi_{t+1} + p_t, & p_t - \lambda p_t = \alpha \lambda y_t + \lambda(1-\lambda) \sum_{k=0}^{\infty} (1-\lambda)^k \mathbb{E}_{t-1-k}(p_t + \alpha y_t), \\
\frac{\pi_t}{\lambda} = -p_{t-1} + \lambda(p_t + \alpha y_t) + (1-\lambda)\mathbb{E}_t\left(\frac{\pi_{t+1}}{\lambda} + p_t\right), & p_t = \frac{\alpha\lambda}{1-\lambda} y_t + \lambda \sum_{k=0}^{\infty} (1-\lambda)^k \mathbb{E}_{t-1-k}(p_t + \alpha y_t), \\
\frac{\pi_t}{\lambda} = (p_t - p_{t-1}) + \lambda \alpha y_t + \frac{1-\lambda}{\lambda}\mathbb{E}_t \pi_{t+1}, & p_{t-1} = \lambda \sum_{k=0}^{\infty} (1-\lambda)^k \mathbb{E}_{t-1-k}(p_{t-1} + \alpha y_{t-1}), \\
\pi_t = \frac{\alpha\lambda^2}{1-\lambda} y_t + \mathbb{E}_t \pi_{t+1}. & \pi_t = \frac{\alpha\lambda}{1-\lambda} y_t + \lambda \sum_{k=0}^{\infty} (1-\lambda)^k \mathbb{E}_{t-1-k}(\pi_t + \alpha \Delta y_t). \\
\pi_t = \beta y_t + \mathbb{E}_t \pi_{t+1}; \qquad \pi_t = \beta y_t + \pi_{t-1}; & \pi_t = \frac{\beta}{\lambda} y_t + \lambda \sum_{j=0}^{\infty} (1-\lambda)^k \mathbb{E}_{t-1-k}(\pi_t + \alpha \Delta y_t).
\end{cases}
$$

$$
AD\begin{cases}
m_t = p_t + y_t; \qquad m_t = p_t + y_t; \qquad m_t = p_t + y_t. \\
p_t^* = (1-\alpha)p_t + \alpha m_t; \qquad p_t^* = (1-\alpha)p_t + \alpha m_t; \qquad p_t^* = (1-\alpha)p_t + \alpha m_t.
\end{cases}
$$

Figure: Two rational expectations and one irrational expectation

Source: Deng, Lecture notes on macroeconomic theory, 2019

# The traditional use of TSMs: forecasting

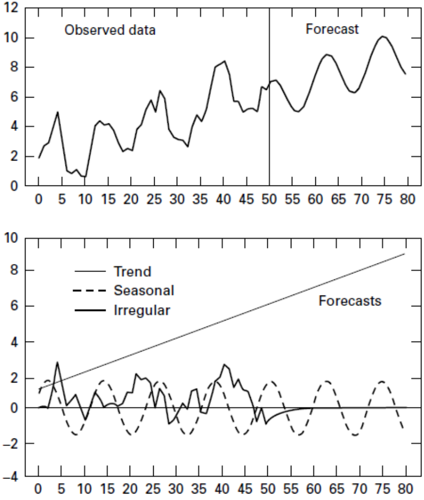


Figure: Hypothetical Time-Series

Source: Enders (2015, p.2)

# Cont'd

If we know

$$y_{t+1} = a_0 + a_1 y_t + \epsilon_{t+1},$$

then

$$\mathbb{E}_t y_{t+1} = a_0 + a_1 y_t,$$

and since

$$y_{t+2} = a_0 + a_1 y_{t+1} + \epsilon_{t+2},$$

then

$$\begin{aligned}\mathbb{E}_t y_{t+2} &= a_0 + a_1 \mathbb{E}_t y_{t+1} \\ &= a_0 + a_1 (a_0 + a_1 y_t) \\ &= a_0 + a_0 a_1 + (a_1)^2 y_t.\end{aligned}$$

The general methodology used to make such forecasts entails finding the equation of motion driving a stochastic process and using that equation to predict subsequent outcomes.

## Cont'd

We can think of a time series as being composed of:

$$
\begin{aligned}
Y_t &= T_t + S_t + I_t, \\
T_t &= 1 + 0.1t, \\
S_t &= 1.6 \sin\left(\frac{\pi}{6}t\right), \\
I_t &= 0.7 I_{t-1} + \epsilon_t.
\end{aligned}
$$

where $T_t$=trend in t (permanent); $S_t$=cycle (e.g.,seasonal component) in t (temporary and predictable). $\Rightarrow$ Both functions of time t.

$I_t$=the irregular disturbance in t; $\epsilon_t$=the random disturbance in t (Noise: unpredictable).

In a most general form, a DE expresses the value of a variable as a function of its own lagged values, time, and other variables.

## Reduced-form equations and structural equations

Consider Samuelson's (1939) classic model:

$$\begin{cases} Y_t & = & C_t + I_t, \\ C_t & = & \alpha Y_{t-1} + \epsilon_{Ct}, \quad 0 < \alpha < 1, \\ I_t & = & \beta(C_t - C_{t-1}) + \epsilon_{It}, \quad \beta > 0. \end{cases}$$

In this Keynesian model, $Y_t, C_t, I_t$ are endogenous variables; $Y_{t-1}, C_{t-1}$ are called predetermined or lagged endogenous variables; $\epsilon_{Ct}, \epsilon_{It}$ are zero mean random disturbances; $\alpha, \beta$ are parameters to be estimated or calibrated.

$$\begin{aligned} I_t &= \beta[(\alpha Y_{t-1} + \epsilon_{Ct}) - C_{t-1}] + \epsilon_{It} \\ &= \beta[(\alpha Y_{t-1} + \epsilon_{Ct}) - (\alpha Y_{t-2} + \epsilon_{C,t-1})] + \epsilon_{It} \\ &= \alpha\beta(Y_{t-1} - Y_{t-2}) + \beta(\epsilon_{Ct} - \epsilon_{Ct-1}) + \epsilon_{It} \\ Y_t &= (\alpha Y_{t-1} + \epsilon_{Ct}) + [\alpha\beta(Y_{t-1} - Y_{t-2}) + \beta(\epsilon_{Ct} - \epsilon_{Ct-1}) + \epsilon_{It}] \\ &= \alpha(1+\beta)Y_{t-1} - \alpha\beta Y_{t-2} + (1+\beta)\epsilon_{Ct} + \epsilon_{It} - \beta\epsilon_{Ct-1} \\ &\equiv \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + w_t. \quad \text{where } w_t = (1+\beta)\epsilon_{Ct} + \epsilon_{It} - \beta\epsilon_{Ct-1} \\ Y_t &= \phi_0 + \sum \phi_i Y_{t-i} + w_t. \quad i = 1 \cdots p, \; w_t : \text{the forcing process} \end{aligned}$$

# The random walk hypothesis

An important special case for the $\{w_t\}$ sequence is

$$w_t = \sum_{i=0}^{\infty} \beta_i \epsilon_{t-i}, \quad \text{where } \beta_i \text{ are constants or 0.}$$

The sequence $\{\epsilon_t\}$ are not functions of the $y_t$ $\rightarrow$ unspecified exogenous shocks, e.g., let $\{\epsilon_t\}$ be a random error term and set $\beta_0 = 1,\ \beta_1 = \beta_2 = \cdots = 0$, then

$$\begin{aligned}
y_t &= \phi_0 + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t, \\
y_t &= 0 + 1 \times y_{t-1} + \epsilon_t, \quad \leftarrow \text{the random walk model} \\
\Delta y_t &= \epsilon_t, \quad \text{a random disturbance term: } \mathbb{E}\epsilon_t = 0 \\
\Delta y_t &= \phi_0 + \phi_1 y_{t-1} + \epsilon_t \text{ the testable restriction: } \phi_0 = \phi_1 = 0.
\end{aligned}$$

# Cont'd

$$y_t - y_{t-1} = \phi_0 + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t - y_{t-1},$$

$$\Rightarrow \Delta y_t = \phi_0 + \gamma y_{t-1} + \sum_{i=2}^{p} \phi_i y_{t-i} + \epsilon_t, \text{ where } \gamma = (\phi_1 - 1).$$

A solution to a difference equation (is a function rather than a number) expresses the value of $y_t$ as a function of the elements of the $\{\epsilon_t\}$ and t and possibly initial conditions.

For example

$$y_t = y_{t-1} + 2 \overset{\text{a solution}}{\Longrightarrow} y_t = 2t + c \rightarrow y_{t-1} = 2(t-1) + c \overset{\text{verify}}{\Longrightarrow} y_t - y_{t-1} = 2.$$

Another example

$$I_t = 0.7 I_{t-1} + \epsilon_t \overset{\text{a solution}}{\Longrightarrow} I_t = \sum_{i=0}^{\infty} 0.7^i \epsilon_{t-i} \rightarrow I_{t-1} = \cdots \overset{\text{verify}}{\Longrightarrow} I_t - 0.7 I_{t-1} = \epsilon_t.$$

Q&A: Reduced-form equations vs. Solutions

# Textbooks

Enders (2015), Applied Econometric Time Series, 4th ed.

Lütkepohl and KRätzig (2004), Applied Time Series Econometrics.

Lütkepohl (2005), New Introduction to Multiple Time Series Analysis.

Cochrane (2005), Time Series for Macroeconomics and Finance.

Hamilton (1994)

MIT-open course (2007)

Chiang (2005, 4th ed.)

R, Matlab, Python; Dynare, JMulti, Eviews.

Time Series Analysis and Its Applications with R example.

Time Series Analysis Using Matlab and R.

Univariate Time Series Analysis with Matlab.

Time Series Analysis with Python.

Lütkepohl (2004, 2005) use JMulti.

# Requirements

60%, final exam.

30%, assignment.

10%, attendence.

## Course Outline

The Mid-Autumn Festival (week 1)

**Lec** 1. 4 Methods to Solve Linear Defference Equations (week 2-3)

1.1 Solving DEs with Constant Coefficients and Constant Terms (Chiang, ch.17; Enders, ch.1)

1.2 Solving DEs with Constant Coefficients and Variable Terms (Enders, ch.1; Hamilton, ch.2, ch.1)

**Lec** 2. Covariance-Stationary ARMA Models (week 4-5)

2.1 Stationary Restrictions for ARMA(p, q) Models (Enders, ch.2; Hamilton, ch.3)

2.2 The Autocorrelation Function (Enders, ch.2)

2.3 ACF+PACF+AIC+SBC → Identification/Specification → Estimation → Diagnostic Check → Forecasting

**Lec** 3. Covariance-Stationary Vector Processes (week 6)

3.1 VAR(p)→VAR(1) (Hamilton, ch.10)

## Cont'd

**Lec** 4. Forecasting/Prediction (week 7)

4.1 Principles of Forecasting (Hamilton, ch.4)

4.2 Predicting ARMA Mdoels (Cochrane, ch.5)

4.3 Wold Decomposition Theorem (Cochrane, ch.6)

**Lec** 5. Parameter Calibration and Impulse Response Simulation

**Lec** 6. Identification and Estimation (week 8) 6.1 ARMA 6.2 VAR

**Lec** 7. Autocovariance-Generating Functions and Spectral Analysis (week 9)

7.1 The Autocovariance-Generating Function for ARMA Models (Hamilton, ch.3)

7.2 The Autocovariance-Generating Function for Vector Processes (Hamilton, ch.10, ch.6)

# Cont'd

**Lec** 8. Modeling Volatility (week 9)

**Lec** 9. Unit Roots (Models with Trend) (week 10)

**Lec** 10. VARs and SVARs (Multiequation Time-Series Models) (week 11-12)

**Lec** 11. Spectral Representation (week 13)

**Lec** 12. Spectral Analysis in Finite Samples (week 14)

**Lec** 13. Cointegration and Error-Correction Models (week 15)

**Lec** 14. Nonlinear Models (Chiang, 2005, ch.17.6, p.562) (week 16)

# Q & A

1) Time series econometrics (TSE) vs. Cross-section econometrics (CSE) ref.

CSE mainly deals with i.i.d.observations, while in TSE each new arriving observation is stochastically depending on the previously observed.

The dependence screw up inferences: the ordinary CLT should be corrected to hold for dependent observations;
The dependence allow us to make forecasts.

2) Continuous Time (Differentials) vs. Discrete Time (Differences)

3) First Order vs. Higher Order (linear vs. nonlinear; homogeneous vs. nonhomogeneous)

4) Univariate vs. Multivariate

5) Stationary vs. Non-Stationary

6) Deterministic Difference Equation (DDE) vs. Stochastic Difference Equation (SDE) (Homoscedasticity vs. Heteroskedasticity)

7) Classics (ARMA, VARMA; unit root, cointegration) vs. DSGE (simulated GMM, ML, Bayesian)

# Learning Objectives

1) The first derivative $\frac{\mathrm{d}y}{\mathrm{d}t}$ is the only one that can appear in a first-order differential equation $\rightarrow \frac{\Delta y}{\Delta t}$, here t can now take only interger values.
$\Delta t = 1 \rightarrow$the 1st-oreder DE $\rightarrow$ comparing the values of y in two consecutive periods.

2) Explain what it means to solve a DE and explain how SDE can be used for forecasting (the role of the general solutions).

3) Demonstrate how to find the solution of DEs with constant coefficients and constant term (the 1st-order, 2nd-order, and pth-orther).

4) Demonstrate how to find the solution of DEs with constant coefficients and variable term (the 1st-order, 2nd-order, and pth-order; deterministic and stochastic)

5) Explain how to use **lag operators** to find the particular solution to a SDE.

6) Estimation $\Rightarrow$ Forecast $\Rightarrow$ Interpreting $\Rightarrow$ Testing hypothesis concerning economic data.

# Definition of 1st-order and higher order DEs

Consider the function

$$y = f(t).$$

If we evaluate the function when the independent variale t takes on the specific value $t^*$.

Using this same notation, $y_{t^*+h}$ represents the value of y when t takes on the specific value $t^* + h$.

Definition

The first difference of y is defined as the value of the function when evaluated at $t = t^* + h$ minus the value of the function evaluated at $t^*$:

$$\begin{aligned} \Delta y_{t^*+h} &\equiv f(t^* + h) - f(t^*) \\ &\equiv y_{t^*+h} - y_{t^*}. \end{aligned}$$

$$h \to 0 \quad : \quad \text{differentials.}$$

Since most economic data is collected over discrete periods, it's more useful to allow the length of the time period $> 0$.

## Cont'd

Using DEs, we normalize units so that h represents a unit change in t (i.e., h=1) and consider the sequence of equally spaced values of the independent variable. We can drop the asterisk (*),

$$
\begin{aligned}
\Delta y_{t+1} &\equiv f(t+1) - f(t) \\
&\equiv y_{t+1} - y_t, \\
\Delta y_{t+2} &\equiv f(t+2) - f(t+1) \\
&\equiv y_{t+2} - y_{t+1}, \\
&\vdots \\
\Delta y_{t+p} &\equiv f(t+p) - f(t+p-1) \equiv y_{t+p} - y_{t+p-1}, \\
\Delta y_t &\equiv f(t) - f(t-1) \equiv y_t - y_{t-1} \quad \Leftarrow \quad p = 0, \\
&\qquad\qquad\qquad \equiv y_{t+1} - y_t, \text{ (see Chiang, 2005, p.545 eq.(17.1))} \\
\Delta^2 y_t &\equiv \Delta(\Delta y_t) = \Delta(y_t - y_{t-1}) = y_t - 2y_{t-1} - (-1)y_{t-2}, \\
&\qquad \Delta(y_{t+1} - y_t) = (y_{t+2} - y_{t+1}) - (y_{t+1} - y_t) = \cdots, \text{ (p.568)} \\
\Delta^p y_t &= \Delta(\cdots(\Delta y_t)) = y_t - a_1 y_{t-1} - \cdots - a_p y_{t-p}.
\end{aligned}
$$

# Solving 1st-order linear DEs with 2 methodologies

$$\dot{y(t)} \equiv \frac{\mathrm{d}y}{\mathrm{d}t} \qquad \text{vs.} \qquad \Delta y_t \equiv \frac{\Delta y_t}{(t+1)-t}.$$

$$\Delta y_t \equiv y_{t+1} - y_t.$$

$$\Delta y_t = 2 \;\Rightarrow\; y_{t+1} = y_t + 2 \;\xrightarrow{\text{iteration}}\; y_t = y_0 + 2t \;\overset{y_0=15}{=\!=\!=}\; 15 + 2t;$$

$$\Delta y_t = -0.1 y_t \;\Rightarrow\; y_{t+1} = 0.9 y_t \;\xrightarrow{\text{iteration}}\; y_t = y_0 \cdot 0.9^t \;\leftarrow\; \text{a homogeneous eq.};$$

$$\Delta y_t = \frac{n-m}{m} y_t \;\Rightarrow\; m y_{t+1} - n y_t = 0 \;\Rightarrow\; y_{t+1} = \frac{n}{m} y_t \;\Rightarrow\; y_t = y_0 \cdot \left(\frac{n}{m}\right)^t \equiv A b^t$$

$$\Delta y_t = \frac{n-1}{1} y_t \;\Rightarrow\; y_{t+1} - n y_t = 0 \;\Rightarrow\; y_{t+1} = \frac{n}{1} y_t \;\Rightarrow\; y_t = y_0 \cdot \left(\frac{n}{1}\right)^t \equiv A n^t;$$

$$\Delta y_t = (b-1) y_t + c \;\Rightarrow\; y_{t+1} - b y_t = c \;\Rightarrow\; y_{t+1} = b y_t + c \;\Rightarrow\; y_t = ?$$

where b and c are two constants. The general solution will consist of the sum of two components: $y_t^h$ (represents the deviations of time path from some intertemporal equilibrium level of y)$+y_t^p$ (represents the intertemporal equilibrium level of y).

## Cont'd

Let $c = 0$,

$$y_{t+1} - by_t = 0 \quad \text{or} \quad y_{t+1} - ay_t = 0,$$

which is a homogeneous equation. Solving it then get $y_t^h = Ab^t$ or $y_t^h = Ab^t$ where $(b = a)$.

When $c \neq 0$,

now let us search for the particular solution $y_t^p$. propose a trial sulution of the simplest form $y_t^p = k$ (a constant), and $y_{t+1}^p = k$, then

$$k = bk + c \quad \Rightarrow \quad y_t^p = k = \frac{c}{1-b}, \quad b \neq 1.$$

If it happens that $b = 1$, $y_t^p$ is not defined. In this event, we employ another form $y_t^p = kt$ and $y_{t+1}^p = k(t+1)$, then

$$k(t+1) = bkt + c \quad \Rightarrow \quad k = \frac{c}{t+1-bt} \overset{b=1}{=\!=\!=} c \quad \Rightarrow \quad y_t^p = kt = ct.$$

This form of $y_t^p$ is a nonconstant function of t and it therefore represents a moving equilibrium.

## Cont'd

The general solution of the first-order difference equation:

$$y_t = \begin{cases} Ab^t + \frac{c}{1-b}, & b \neq 1; \\ Ab^t + ct = A + ct, & b = 1. \end{cases}$$

To eliminate the arbitrary constant A, we resort to an initial condition that $y_t = y_0$.

$$y_0 = \begin{cases} A + \frac{c}{1-b}, & b \neq 1; \\ A + ct = A, & b = 1. \end{cases} \quad \Rightarrow \quad A = \begin{cases} y_0 - \frac{c}{1-b}, & b \neq 1; \\ y_0, & b = 1. \end{cases}$$

The definite solution of the first-order difference equation:

$$y_t = \begin{cases} \left(y_0 - \frac{c}{1-b}\right) b^t + \frac{c}{1-b}, & b \neq 1; \\ y_0 b^t + ct = y_0 + ct, & b = 1. \end{cases}$$

The dynamic stability of equilibrium depends on the $Ab^t$ term as $t \to \infty$.

# Cont'd: The dynamic stability of equilibrium

Case 1. The role of b. Assuming $A = 1$, the term becomes $b^t$ and the stability depends only on the significance of b.

| | | | | Value of $b^t$ in Different Time Periods | | | | |
|---|---|---|---|---|---|---|---|---|
| Region | Value of $b$ | | Value of $b^t$ | $t=0$ | $t=1$ | $t=2$ | $t=3$ | $t=4 \cdots$ |
| I | $b > 1$ | $(\|b\| > 1)$ | e.g., $(2)^t$ | 1 | 2 | 4 | 8 | 16 |
| II | $b = 1$ | $(\|b\| = 1)$ | $(1)^t$ | 1 | 1 | 1 | 1 | 1 |
| III | $0 < b < 1$ | $(\|b\| < 1)$ | e.g., $\left(\frac{1}{2}\right)^t$ | 1 | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{8}$ | $\frac{1}{16}$ |
| IV | $b = 0$ | $(\|b\| = 0)$ | $(0)^t$ | 0 | 0 | 0 | 0 | 0 |
| V | $-1 < b < 0$ | $(\|b\| < 1)$ | e.g., $\left(-\frac{1}{2}\right)^t$ | 1 | $-\frac{1}{2}$ | $\frac{1}{4}$ | $-\frac{1}{8}$ | $\frac{1}{16}$ |
| VI | $b = -1$ | $(\|b\| = 1)$ | $(-1)^t$ | 1 | $-1$ | 1 | $-1$ | 1 |
| VII | $b < -1$ | $(\|b\| > 1)$ | e.g., $(-2)^t$ | 1 | $-2$ | 4 | $-8$ | 16 |

Figure: A Classification of the Values of b

Source: Chiang (2005, p.552)

# Cont'd

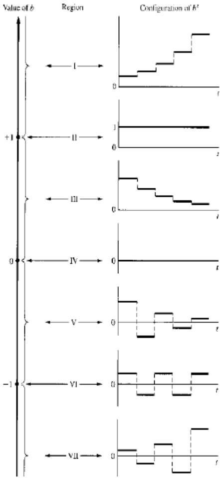


Figure: A Classification of the Values of b

Source: Chiang (2005, p.553)

## Cont'd

The essence of the preceding discussion can be conveyed in the following general statement: The time path of $b^t (b \neq 0)$ will be

$$\left.\begin{array}{l}\text{Oscillatory}\\ \text{Nonoscillatory}\end{array}\right\} \quad \text{if} \quad \begin{cases} b < 0 \\ b > 0. \end{cases}$$

$$\left.\begin{array}{l}\text{Convergent}\\ \text{Divergent}\end{array}\right\} \quad \text{if} \quad \begin{cases} |b| < 1 \\ |b| \geq 1. \end{cases}$$

Note that the convergence of the $b^t$ expression hinges on the absolute value of b.

Case 2. The role of A. First, the magnitude of A can serve to "blow up" (if, say, $A = 2$) or "pare down" (if, say, $A = \frac{1}{2}$) the values of $b^t$. That is, it can produce a scale effect without changing the basic configuration of the time path. However, a negative A (if, say, $A = -2$) will produce a mirror effect as well as a scale effect.

## Cont'd: Convergence to equilibrium

The solution to a homogeneous equation ($Ab^t \leftrightarrow A\lambda^t$), as we recall, represents the deviations from some intertemporal equilibrium level. If a particular solution (if, say, $y_t^p = 5$) is added to the $Ab^t$ term, the time path must be shifted up vertically by a constant value of 5. This will in no way affect the convergence or divergence of the time path, but it will alter the level with reference to which convergence or divergence in gauged.

What the above figure is the convergence or divergence of the $y_t^h = Ab^t$ to 0. When the $y_p$ is included, it becomes the convergence or divergence of $y_t = y_t^h + y_t^p$ to the equilibrium level $y_t^p$.

When $b = 1$, $y_t = y_t^h + y_t^p = Ab^t + y_t^p = A + y_t^p = y_0 + ct$, it can never reach $y_t^p$ unless $A = 0$ or $y_0 = 0$. Recall that $y_t^p = ct$ is a moving equilibrium. With a nonzero A or $y_0$, there will be a constant (A or $y_0$) deviation from the moving equilibrium, thus, this time path is to be considered divergent.

# Solving 2nd-order linear DEs

A simple variety of second-order linear and nonhomogeneous difference equations with constant coefficients and constant term takes the form

$$y_{t+2} = a_1 y_{t+1} + a_2 y_t + c \quad \text{vs.} \quad y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + w_t.$$
$$y_t = y_t^p + y_t^h.$$

The particular solution ($y_t^p$) defined as any solution to the complete difference equation. where $w_t$ will be regarded as constant parameters $\{w_0, w_1, w_2, \ldots\}$ or a random variable.

Try a solution of the form $y_t^p = k$ when $a_1 + a_2 \neq 1$

$$k - a_1 k - a_2 k = c \Rightarrow y_t^p = k = \frac{c}{1 - a_1 - a_2}, \text{ where } a_1 + a_2 \neq 1;$$

and of the form

$$\begin{cases} y_t^p = kt & \text{when } a_1 + a_2 = 1 (\text{but at the same time } a_1 \neq 2; \\ y_t^p = kt^2 & \text{when } a_1 + a_2 = 1 (\text{but at the same time } a_1 = 2, a_2 = -1). \end{cases}$$

# Cont'd

$$
\begin{aligned}
& k(t+2) - a_1 k(t+1) - a_2 kt = c \\
\Rightarrow \quad & k = \frac{c}{(t+2) - a_1(t+1) - a_2 t} = \frac{c}{2 - a_1} \\
\Rightarrow \quad & y_t^p = kt = \frac{c}{2 - a_1} t \quad \text{case of } a_1 \neq 2, a_1 + a_2 = 1;
\end{aligned}
$$

$$
\begin{aligned}
& k(t+2)^2 - a_1 k(t+1)^2 - a_2 kt^2 = c \\
\Rightarrow \quad & k = \frac{c}{(t+2)^2 - 2(t+1)^2 + t^2} = \frac{c}{2} \\
\Rightarrow \quad & y_t^p = kt^2 = \frac{c}{2} t^2 \quad \text{case of } a_1 = 2, a_1 + a_2 = 1.
\end{aligned}
$$

# Cont'd

$y_t^h$ corresponding to the homogeneous equation

$$y_{t+2} - a_1 y_{t+1} - a_2 y_t = 0 \quad \text{vs.} \quad y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} = 0.$$

Try a solution of the form $y_t^h = Ab^t$, thus

$$Ab^{t+2} - a_1 Ab^{t+1} - a_2 Ab^t = 0, \quad \text{vs.} \quad A\lambda^t - \phi_1 A\lambda^{t-1} - \phi_2 A\lambda^{t-2} = 0.$$

$$\xrightarrow{\text{divided by } Ab^t} b^2 - a_1 b - a_2 = 0 \leftarrow \text{ the characteristic equation,}$$

$$\Rightarrow \text{characteristic roots } b_1, b_2 = \frac{a_1 \pm \sqrt{a_1^2 + 4a_2}}{2} \equiv \frac{a_1 \pm \sqrt{d}}{2}.$$

$$\xrightarrow{\text{divided by } A\lambda^{t-2}} \lambda^2 - \phi_1 \lambda - \phi_2 = 0 \leftarrow \text{ the characteristic equation,}$$

$$\Rightarrow \text{characteristic roots } \lambda_1, \lambda_2 = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2}.$$

Each of $b_1, b_2$ is acceptable in the solution $Ab^t$.

## Cont'd

Three possible situations may be encountered in regard to the characteristic roots, depending on d.

**Case 1** (distinct real roots: $d > 0$, i.e., $a_1^2 + 4a_2 > 0$), $b_1, b_2$ are real and distinct,

$$
\begin{aligned}
b_1, b_2 &= \frac{a_1 \pm \sqrt{a_1^2 + 4a_2}}{2}, \\
y_t^h &= A_1 b_1^t + A_2 b_2^t, \\
y_t^p &= \frac{c}{1 - a_1 - a_2} \quad \text{or} \quad \frac{c}{2 - a_1} t \quad \text{or} \quad \frac{c}{2} t^2, \\
y_t &= \begin{cases} y_t^h + y_t^p = A_1 b_1^t + A_2 b_2^t + \frac{c}{1-a_1-a_2}, & a_1 + a_2 \neq 1; \\ y_t^h + y_t^p = A_1 b_1^t + A_2 b_2^t + \frac{c}{2-a_1} t, & a_1 + a_2 = 1, a_1 \neq 2; \\ y_t^h + y_t^p = A_1 b_1^t + A_2 b_2^t + \frac{c}{2} t^2, & a_1 + a_2 = 1, a_1 = 2. \end{cases}
\end{aligned}
$$

Suppose given $y_0$ and $y_1$. Then we can find out $A_1$ and $A_2$ by letting $t = 0, 1$.

# Cont'd

**Case 2** (repeated real roots: $d = 0$, i.e., $a_1^2 + 4a_2 = 0$)

$$b = b_1 = b_2 = a_1/2,$$
$$A_1 b_1^t + A_2 b_2^t = (A_1 + A_2)b^t \equiv A_3 b^t \rightarrow \text{short of one constant},$$
$$y_t^h = A_3 b^t + A_4 t b^t.$$

Given two initial conditions, $A_3$ and $A_4$ can again be assigned definite values.

**Case 3** (complex roots: $d < 0$, i.e., $a_1^2 + 4a_2 < 0$) The characteristic roots which are conjugate complex will be in the form

$$\begin{aligned}
b_1, b_2 &= \frac{a_1}{2} \pm \frac{\sqrt{-(a_1^2 + 4a_2)}i}{2} \equiv h \pm vi, \\
y_t^h &= A_1 b_1^t + A_2 b_2^t \\
&= A_1 (h + vi)^t + A_2 (h - vi)^t \\
&= A_1 R^t (\cos\theta t + i \sin\theta t) + A_2 R^t (\cos\theta t - i \sin\theta t) \\
&= R^t [(A_1 + A_2)\cos\theta t + (A_1 - A_2) i \sin\theta t] \\
&\equiv R^t (A_5 \cos\theta t + A_6 \sin\theta t).
\end{aligned}$$

# Cont'd: De Moivre's theorem

Hamilton (1994, p.14)

$$
\begin{aligned}
b_1 &= h + vi = Re^{i\theta} = R(\cos\theta + i\sin\theta),\\
b_2 &= h - vi = Re^{-i\theta} = R(\cos\theta - i\sin\theta);\\
b_1^p &= (h + vi)^p = R^p e^{i\theta p} = R^p[\cos(\theta p) + i\sin(\theta p)],\\
b_2^p &= (h - vi)^p = R^p e^{-i\theta p} = R^p[\cos(\theta p) - i\sin(\theta p)];
\end{aligned}
$$

$$b_1^t, b_2^t = (h \pm vi)^t = R^t(\cos\theta t \pm i\sin\theta t), \quad \leftarrow \text{see Hamilton (1994, p.708)}$$

$$\text{where } R = \sqrt{h^2 + v^2} = \sqrt{\frac{a_1^2 - (a_1^2 + 4a_2)}{4}} = \sqrt{-a_2},$$

$$\cos\theta = \frac{h}{R} = \frac{a_1}{2\sqrt{-a_2}} \quad \text{and} \quad \sin\theta = \frac{v}{R} = \sqrt{1 + \frac{a_1^2}{4a_2}}.$$

# Cont'd: The convergence of the time path

As in the case of 1st-order DEs, the convergence of the time path $y_t$ hinges solely on whether $y_t^h$ tends toward 0 as $t \to \infty$. The various configurations of $b^t$ is still applicable, however, we shall have to consider 2 characterics roots $b_1, b_2$ with $A_1, A_2 (d > 0); A_3, A_4 (d = 0); A_5, A_6 (d < 0)$ rather than only $b$.

Case 1. The role of b (1st-order DEs) and $b_1, b_2$ (2nd-order DEs)

Case 2. The role of A (1st-order DEs) and $A_1 - A_6$ (2nd-order DEs)

(1) $d > 0, b_1 \neq b_2 \to y_t^h = A_1 b_1^t + A_2 b_2^t$ (Assume $A_1, A_2 = 1$)

The time path of $b_1^t, b_2^t (b_1, b_2 \neq 0)$ will be

$$\left.\begin{array}{l}\text{Convergent}\\ \text{Divergent}\end{array}\right\} \quad \text{if} \quad \begin{cases}|b_1| < 1, |b_2| < 1;\ |b_1| > 1, |b_2| < 1 \text{ if } |b_2| > |b_1|;\\ |b_1| > 1, |b_2| > 1;\ |b_1| > 1, |b_2| < 1 \text{ if } |b_1| < |b_2|.\end{cases}$$

Note that even though the eventual convergence depends on the dominant root (with the higher absolute value) alone, the nondominant root will exert a definite influence on the time path, too, at least in the beginning periods. Therefore, the exact configuration of $y_t$ is still dependent on both roots.

## Cont'd

(2) $d = 0, b_1 = b_2 = b \to y_t^h = A_3 b^t + A_4 t b^t$ (Assume $A_3, A_4 = 1$)

The time path of $b^t (b \neq 0)$ will be

$$\left.\begin{array}{l}\text{Convergent}\\ \text{Divergent}\end{array}\right\} \quad \text{if} \quad \begin{cases} |b| < 1; \\ |b| > 1. \end{cases}$$

If $|b| > 1$, the t part will simply serve to intensify the explossiveness as t increases; If $|b| < 1$, the t part will run counter to each other, but the damping force of $b^t$ will always win over the exploding force of t.

(3) $d < 0, b_1 \neq b_2 \to y_t^h = R^t (A_5 \cos\theta t + A_6 \sin\theta t)$ (Assume $A_5, A_6 = 1$)

As far as convergence is concerned, the decisive factor is really the $R^t$ term, wich will dictate whether the stepped fluctuation is to be intensified or mitigated as t increases. In the present case, the fluctuation can be gradually narrowed down if and only if (iff) $R < 1$. Since R is by definition the absolute value of the conjugate complex roots $(h \pm vi)$, the condition for convergence is again that the characteristic roots be less than unity in absolute value.

## Cont'd

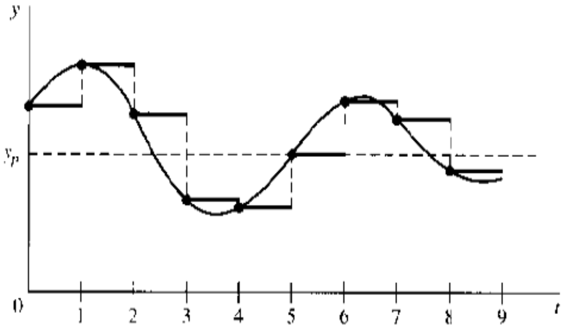


Figure: The Time Path of y for the Complex-Root Case

**To summarize**: For all three cases of characteristic roots, the time path will converge to a (stationary or moving) intertemporal equilibrium ($y_t^p$)—regardless of what the initial conditions may happen to be—iff the absolute value of every root is less than 1.

Source: Chiang (2005, p.575)

# Solving pth-order linear DEs

A pth-order linear DE with constant coefficients and constant term may be written as

$y_{t+p} = a_1 y_{t+p-1} + \cdots + a_{p-1} y_{t+1} + a_p y_t + c;$

$y_t = \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + w_t$ Chiang p.586, variable-term and higher-order

The particular solution $y_t^p = k$ or $y_t^p = kt$ or $y_t^p = kt^2$, etc., try in that order.

# Cont'd

$y_t^h$ corresponding to the homogeneous function

$$y_{t+p} = a_1 y_{t+p-1} + \cdots + a_{p-1} y_{t+1} + a_p y_t;$$
$$y_t = \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p}.$$
$$y_t = y_t^p + y_t^h.$$

Try a solution of the form $y_t^h = Ab^t$, thus

$$Ab^{t+p} - a_1 Ab^{t+p-1} + \cdots - a_{p-1} Ab^{t+1} - a_p Ab^t = 0,$$
$$A\lambda^t - \phi_1 A\lambda^{t-1} - \cdots - \phi_p A\lambda^{t-p} = 0.$$

$\xrightarrow{\text{divided by } Ab^t} b^p - a_1 b^{p-1} - \cdots - a_{p-1} b - a_p = 0 \leftarrow$ a characteristic equation

$\Rightarrow$ characteristic roots $b_1, b_2, \ldots, b_p = b_i (i = 1, 2, \ldots, p) =?$

$\xrightarrow{\text{divided by } A\lambda^{t-p}} \lambda^p - \phi_1 \lambda^{p-1} - \cdots - \phi_{p-1}\lambda - \phi_p = 0 \leftarrow$ a characteristic equation

$\Rightarrow$ characteristic roots $\lambda_1, \lambda_2, \ldots, \lambda_p = \lambda_i (i = 1, 2, \ldots, p) =?$

# Cont'd

all of which should enter into the solution to the homogeneous function (i.e., the complementary function) thus:

**Case 1** (distinct real roots)

$$y_t^h = \sum_{i=1}^{p} A_i b_i^t.$$

**Case 2** (repeated real roots)

$$y_t^h = A_1 b_1^t + A_2 b_2^t + \cdots + A_p b_p^t = (A_1 + A_2 + \cdots + A_p) b^t = A_{p+1} b^t \to \text{short of}$$
$$y_t^h = A_{p+1} b^t + A_{p+2} t b^t + A_{p+3} t^2 b^t + \cdots + A_{2p} t^{p-1} b^t.$$

**Case 3** (complex roots) If there is a pair of conjugate complex roots (say, $b_{p-1}, b_p$) then the last two terms in the sum are to be combined into the expression

$$R^t (A_{2p+1} \cos \theta t + A_{2p+2} \sin \theta t).$$

A similar expression can also be assigned to any other pair of complex roots. In case of two repeated pairs of complex roots, however, one of the two must be given a multiplicative factor of $tR^t$ instead of $R^t$.

# Cont'd: The Schur theorem and convergence

Recall that the time path can converge iff every root of the characteristic eq. is less than 1 in absolute value. In view of this, the Schur theorem becomes directly applicable: The roots of the pth-degree polynomial eq. (i.e., the characteristic eq.) will al be less than unity in absolute value iff the following p determinants are all positive,

$$\Delta_1 = \begin{vmatrix} a_0 & a_n \\ a_n & a_0 \end{vmatrix} \qquad \Delta_2 = \begin{vmatrix} a_0 & 0 & a_n & a_{n-1} \\ a_1 & a_0 & 0 & a_n \\ a_n & 0 & a_0 & a_1 \\ a_{n-1} & a_n & 0 & a_0 \end{vmatrix}$$

$$\Delta_n = \begin{vmatrix} a_0 & 0 & \cdots & 0 & a_n & a_{n-1} & \cdots & a_1 \\ a_1 & a_0 & \cdots & 0 & 0 & a_n & \cdots & a_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1} & a_{n-2} & \cdots & a_0 & 0 & 0 & \cdots & a_n \\ a_n & 0 & \cdots & 0 & a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_n & \cdots & 0 & 0 & a_0 & \cdots & a_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_1 & a_2 & \cdots & a_n & 0 & 0 & \cdots & a_0 \end{vmatrix}$$

Figure: The p (or "n" in the book) determinants

Source: Chiang (2005, p.590). For a discussion of this theorm and its history, see Chipman (1951, p.119)

# Samuelson (1939, RES)

Paul A. Samuelson, "Interactions between the Multiplier Analysis and the Principle Acceleration," Review of Economic Statistics, May 1939, pp.75-78. (Chiang, 2005, p.576)

$$\text{A 1st-order DE} \begin{cases} Y_t = C_t + G_0, & \\ C_t = \alpha Y_{t-1} & \alpha \in (0,1). \end{cases}$$

$$\text{A 2nd-order DE} \begin{cases} Y_t = C_t + I_t + G_0, & \\ C_t = \alpha Y_{t-1} & \alpha \in (0,1), \\ I_t = \beta(C_t - C_{t-1}) & \beta \in (0,\infty), \end{cases}$$

$$\begin{aligned} \Rightarrow \quad & I_t = \beta(\alpha Y_{t-1} - \alpha Y_{t-2}) = \alpha\beta(Y_{t-1} - Y_{t-2}), \\ \Rightarrow \quad & Y_t = \alpha Y_{t-1} + \alpha\beta(Y_{t-1} - Y_{t-2}) + G_0, \\ \Rightarrow \quad & Y_t = (\alpha + \alpha\beta) Y_{t-1} - \alpha\beta Y_{t-2} + G_0, \\ \Rightarrow \; & Y_{t+2} = (\alpha + \alpha\beta) Y_{t+1} - \alpha\beta Y_t + G_0. \end{aligned}$$

This is a 2nd-order linear DE with constant coefficients and constant term.

## Cont'd: The particular solution

Try $Y_t^p = k$

$$k - \alpha(1+\beta)k + \alpha\beta k = G_0,$$
$$\Rightarrow Y_t^p = k = \frac{G_0}{1-\alpha-\alpha\beta+\alpha\beta} = \frac{G_0}{1-\alpha}, \quad \alpha \in (0,1)$$

The $\frac{G_0}{1-\alpha}$ should give us the equilibrium income $Y^*$ in the sense that this income level satisfies the equilibrium condition (national income = total expenditure). Being the particular solution of the model, it also give us the intertemporal equilibrium income $Y_t^*$.

## Cont'd: The homogeneous solution

Resort to the complementary function

$$Y_{t+2} - (\alpha + \alpha\beta)Y_{t+1} + \alpha\beta Y_t = 0.$$

Try $Y_t^h = Ab^t$

$$Ab^{t+2} - (\alpha + \alpha\beta)Ab^{t+1} + \alpha\beta Ab^t = 0.$$

Divide by $Ab^t$

$$b^2 - (\alpha + \alpha\beta)b + \alpha\beta = 0.$$

Characteristic roots

$$b_1, b_2 = \frac{(\alpha + \alpha\beta) \pm \sqrt{(\alpha + \alpha\beta)^2 - 4\alpha\beta}}{2}.$$

The discriminant

$$d = \sqrt{(\alpha + \alpha\beta)^2 - 4\alpha\beta}.$$

$$\left.\begin{array}{ll} \text{Case 1} & d > 0 \\ \text{Case 2} & d = 0 \\ \text{Case 3} & d < 0 \end{array}\right\} \xrightarrow{\text{stable conditions}} |b_1|, |b_2| < 1 \rightarrow \alpha, \beta \rightarrow \alpha\beta < 1.$$

# Cont'd: Case 1

$$\left.\begin{array}{c} b_1 + b_2 = \alpha + \alpha\beta > 0 \\ b_1 b_2 = \alpha\beta > 0 \end{array}\right\} \Rightarrow b_1, b_2 > 0.$$

$$\begin{aligned} \Rightarrow (1 - b_1)(1 - b_2) &= 1 - (b_1 + b_2) + b_1 b_2 \\ &= 1 - (\alpha + \alpha\beta) + \alpha\beta \\ &= 1 - \alpha. \end{aligned}$$

$$\xRightarrow{\alpha \in (0,1)} 0 < (1 - b_1)(1 - b_2) < 1.$$

$$\left.\begin{array}{r} b_1, b_2 > 0 \\ 0 < (1 - b_1)(1 - b_2) < 1 \end{array}\right\} \Rightarrow \begin{cases} (i) & 0 < b_2 < b_1 < 1 & \Rightarrow \\ (ii) & 0 < b_2 < b_1 = 1 & \Rightarrow \\ (iii) & 0 < b_2 < 1 < b_1 & \Rightarrow \\ (iv) & 1 = b_2 < b_1 & \Rightarrow \\ (v) & 1 < b_2 < b_2 & \Rightarrow \end{cases}$$

## Cont'd: Case 2 and 3

The roots are now

$$b_1 = b_2 = b = \frac{\alpha + \alpha\beta}{2}.$$

$$\left.\begin{array}{r} b > 0 \\ 0 < (1-b)^2 < 1 \end{array}\right\} \Rightarrow \begin{cases} (vi) & b < 1 \quad \Rightarrow \\ (vii) & b = 1 \quad \Rightarrow \\ (viii) & b > 1 \quad \Rightarrow \end{cases}$$

With complex roots, we have stepped fluctuation, and hence endogenous business cycles.

$$R = \sqrt{\alpha\beta} \quad \Rightarrow \quad \begin{cases} (ix) & R < 1 \quad \Rightarrow \\ (x) & R = 1 \quad \Rightarrow \\ (xi) & R > 1 \quad \Rightarrow \end{cases}$$

# Cont'd

| Case | Subcase | Values of $\alpha$ and $\gamma$ | Time Path $Y_t$ |
|---|---|---|---|
| 1. Distinct real roots | | | |
| $\gamma > \frac{4\alpha}{(1+\alpha)^2}$ | 1C: $0 < b_2 < b_1 < 1$ | $\alpha\gamma < 1$ | Nonoscillatory and nonfluctuating |
| | 1D: $1 < b_2 < b_1$ | $\alpha\gamma > 1$ | |
| 2. Repeated real roots | | | |
| $\gamma = \frac{4\alpha}{(1+\alpha)^2}$ | 2C: $0 < b < 1$ | $\alpha\gamma < 1$ | Nonoscillatory and nonfluctuating |
| | 2D: $b > 1$ | $\alpha\gamma > 1$ | |
| 3. Complex roots | | | |
| $\gamma < \frac{4\alpha}{(1+\alpha)^2}$ | 3C: $R < 1$ | $\alpha\gamma < 1$ | With stepped fluctuation |
| | 3D: $R \geq 1$ | $\alpha\gamma \geq 1$ | |

Figure: Cases and Subcases of the Samuelson Model

Notice that we have used different symbols in the slides.

Source: Chiang (2005, p.579)

# Cont'd

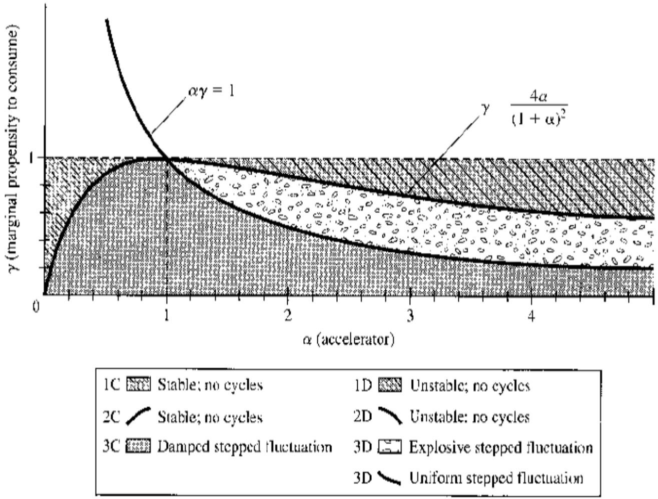


Figure: Cases and Subcases of the Samuelson Model

Source: Chiang (2005, p.577)

## Solution by iteration (recursive substitution)

Cumbersome and time-intensive but intuitive. Our dynamic problem is to find a time path from some given pattern of change of a variable y over time.

$$
\begin{aligned}
& y_t = \phi y_{t-1} + \epsilon_t, \\
& y_t = c + \epsilon_t + \theta \epsilon_{t-1}, \quad \text{MA(1)} \\
& y_t = c + \phi y_{t-1} + \epsilon_t, \quad \text{AR(1)} \\
& y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t, \quad \text{AR(2)} \\
& y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}, \quad \text{ARMA(2, 2)} \\
& \vdots
\end{aligned}
$$

The form of the variable term $\epsilon_t$ (i.e., forcing process) can be very general: it can be any function of time, current and lagged values of other variables, and/or stochastic disturbances.

## Cont'd

**Case 1**. Iteration with an initial condition ($y_0$)

| Data | Equation |
|---|---|
| 0 | $y_0$ given |
| 1 | $y_1 = \phi y_0 + \epsilon_1$ |
| $\vdots$ | $\vdots$ |
| t | $y_t = \phi y_{t-1} + \epsilon_t$ |

$$
\begin{aligned}
y_2 &= \phi y_1 + \epsilon_2, \\
&= \phi(\phi y_0 + \epsilon_1) + \epsilon_2, \\
&= \phi^2 y_0 + \phi \epsilon_1 + \epsilon_2; \\
y_3 &= \phi y_2 + \epsilon_3 \\
&= \phi(\phi^2 y_0 + \phi \epsilon_1 + \epsilon_2) + \epsilon_3, \\
&= \phi^3 y_0 + \phi^2 \epsilon_1 + \phi \epsilon_2 + \epsilon_3. \\
y_t &= \phi^t y_0 + \sum \phi^i \epsilon_{t-i}, \quad i = 0, 1 \ldots, t-1.
\end{aligned}
$$

# Cont'd

**Case 2**. Iteration without an initial condition $(y_0)$

Table: The dynamic equation governs the behavior of y for all t

| Data | Equation |
|---|---|
| 0 | $y_0 = \phi y_{-1} + \epsilon_0$ |
| 1 | $y_1 = \phi y_0 + \epsilon_1$ |
| $\vdots$ | $\vdots$ |
| t | $y_t = \phi y_{t-1} + \epsilon_t$ |

$$y_0 = \text{ given } \epsilon_0 \text{ and } y_{-1}, \qquad y_1 = \text{ given } \epsilon_1,$$

$$y_2 = \text{ given } \epsilon_2,$$

$$\vdots$$

$$\begin{aligned} y_t &= \phi^{t+1} y_{-1} + \phi^t \epsilon_0 + \phi^{t-1}\epsilon_1 + \phi^{t-2}\epsilon_2 + \cdots + \phi \epsilon_{t-1} + \epsilon_t, \\ &= \phi^{t+1} y_{-1} + \sum_{i=0}^{t} \phi^i \epsilon_{t-i} \overset{\text{another p periods}}{=} \phi^{t+p+1} y_{-p-1} + \sum_{i=0}^{t+p} \phi^i \epsilon_{t-i}. \end{aligned}$$

# Cont'd: Dynamic multipliers

Q: What are the effects on y of changes in the value of $\epsilon_t$ given the dynamic system:

$$y_{t+h} = \phi^{h+1} y_{t-1} + \phi^h \epsilon_t + \phi^{h-1}\epsilon_{t+1} + \phi^{h-2}\epsilon_{t+2} + \cdots + \phi\epsilon_{t+h-1} + \epsilon_{t+h},$$

$$= \phi^{h+1} y_{t-1} + \sum_{i=0}^{h} \phi^{h-i}\epsilon_{t+i} \overset{|\phi|<1}{=} 0 + \sum_{i=0}^{\infty} \phi^{h-i}\epsilon_{t+i} = A\phi^{t+h} + \sum_{i=0}^{\infty} \phi^{h-i}\epsilon_{t+i}.$$

$$\phi^t = \frac{\partial y_t}{\partial \epsilon_0} \iff \frac{\partial y_{t+h}}{\partial \epsilon_t} = \phi^h.$$

cf. Enders(2015, p.11)

## Cont'd: An example

Goldfeld (1973) estimated money demand fucntion for the USA.

$$m_t = 0.27 + 0.72m_{t-1} + 0.19I_t - 0.045r_{bt} - 0.019r_{ct};$$
$$\underbrace{m_t}_{y_t} = \underbrace{0.72m_{t-1}}_{\phi y_{t-1}} + \underbrace{0.27 + 0.19I_t - 0.045r_{bt} - 0.019r_{ct}}_{\epsilon_t}.$$

Suppose we want to know what will happen to money demand two quarters from now if current income $I_t$ were to increase by one unit dotay with future income $I_{t+1}$ and $I_{t+2}$ unaffected (with an initial condition $y_{t-1}$):

$$y_{t+2} = \phi^3 y_{t-1} + \phi^2\epsilon_t + \phi\epsilon_{t+1} + \epsilon_{t+2}.$$
$$\frac{\partial m_{t+2}}{\partial I_t} = \frac{\partial m_{t+2}}{\partial \epsilon_t} \times \frac{\partial \epsilon_t}{\partial I_t} = \phi^2 \times \frac{\partial \epsilon_t}{\partial I_t} = (0.72)^2(0.19) = 0.098.$$

Thus, an increase in income $I_t$ of 1% units will result an increase in money holdings $m_t$ of 0.1% units which derive from

$$(0.01) \times (0.098) = 0.001.$$

## Cont'd: Tips

$$\frac{X_t - X_{t-1}}{X_{t-1}} = \frac{X_t}{X_{t-1}} - 1 \quad \Rightarrow$$

$$\frac{X_t}{X_{t-1}} = 1 + \frac{X_t - X_{t-1}}{X_{t-1}} \quad \Rightarrow$$

$$\ln\left(\frac{X_t}{X_{t-1}}\right) = \ln\left(1 + \frac{X_t - X_{t-1}}{X_{t-1}}\right) \approx \frac{X_t - X_{t-1}}{X_{t-1}} \quad \Rightarrow$$

$$\ln X_t - \ln X_{t-1} = \frac{X_t - X_{t-1}}{X_{t-1}};$$

$$\mathrm{d} \ln X(t) = \frac{\mathrm{d}X(t)}{X(t)};$$

$$\hat{x}_t \equiv x_t - x \equiv \ln X_t - \ln X \approx \ln X + \frac{1}{X}(X_t - X) - \ln X = \frac{X_t - X}{X}.$$

# Cont'd

Reconciling the two iterative methods:

$$y_t = A\phi^t + \sum_{i=0}^{\infty} \phi^i \epsilon_{t-i} \quad \text{without an initial condition}$$

$$y_0 = A + \sum_{i=0}^{\infty} \phi^i \epsilon_{-i} \;\Rightarrow\; A = y_0 - \sum_{i=0}^{\infty} \phi^i \epsilon_{-i} \quad y_0 \text{ given}$$

$$\begin{aligned}
\Rightarrow y_t &= \left( y_0 - \sum_{i=0}^{\infty} \phi^i \epsilon_{-i} \right) \phi^t + \sum_{i=0}^{\infty} \phi^i \epsilon_{t-i} \\
&= \phi^t y_0 - \sum_{i=0}^{\infty} \phi^{t+i} \epsilon_{-i} + \sum_{i=0}^{\infty} \phi^i \epsilon_{t-i} \\
&= \phi^t y_0 + \sum_{i=0}^{t-1} \phi^i \epsilon_{t-i}.
\end{aligned}$$

If $|\phi| > 1, p \to \infty \quad \Rightarrow \quad |\phi|^{t+p} \to \infty$. However, if there is an initial condition, there is no need to obtain the infinite summation. Still,

$$y_t = \phi^t y_0 + \sum \phi^i \epsilon_{t-i}, \quad i = 0, \cdots, t-1.$$

## **Cont'd:** $\phi = 1$

The random walk model ($\phi = 1 \;\Rightarrow\; y_t = y_{t-1} + \epsilon_t$)

$$
\begin{aligned}
y_0 &= \text{ given}, \\
y_1 &= y_0 + \epsilon_1, \\
y_2 &= (y_0 + \epsilon_1) + \epsilon_2 = y_0 + (\epsilon_1 + \epsilon_2), \\
y_3 &= [(y_0 + \epsilon_1) + \epsilon_2] + \epsilon_3 = y_0 + (\epsilon_1 + \epsilon_2 + \epsilon_3), \\
&\vdots \\
y_t &= y_0 + \sum_{i=1}^{t} \epsilon_i \qquad \text{vs.} \qquad y_t = \sum_{i=0}^{\infty} \phi^i \epsilon_{t-i}
\end{aligned}
$$

Notice that the solution contains summation of all disturbances from $\epsilon_1 \to \epsilon_t$. Thus, when $\phi = 1$, each disturbance ($\epsilon_i$) has a permanent non-decaying effect on the value of $y_t$.

We should compare the case (without an initial condition but the period $p \to \infty$) where $|\phi| < 1$, $|\phi|^t$ is a decreasing function of t so that the effects of past disturbances become successively smaller over time.

# Cont'd: The magnitude of $\phi$

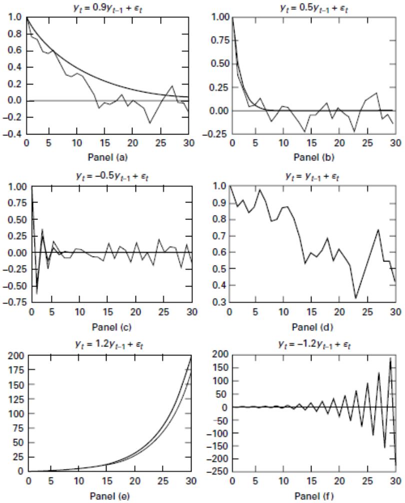


Figure: Convergent and Nonconvergent Sequences of $y_t = \phi^t \times 1 + \sum_{i=0}^{30} \phi^i \epsilon_{t-i}$

Source: Enders (2015, p.13)

# An alternative solution methodology

The general solution $(y_t)$ = All homogeneous solutions $(y_t^h)$ + a particular solution $(y_t^p)$.

Consider only the homogeneous portion of the first-order equation

$$y_t - \phi y_{t-1} = 0 \xlongequal{\text{the homogeneous solution}} y_t^h = A\phi^t y_0 \text{ or } y_t^h = y_{t-1} = \cdots = 0.$$

Once the general solution is obtained, the arbitrary constant A can be eliminated by imposing an initial condition for $y_0$.

STEP 1: find all p homogeneous solutions;

STEP 2: find a particular solution;

STEP 3: obtain the general solution;

STEP 4: eleminate A.

# Cont'd: Generalizing the method

Consider the homogeneous part of a p-th order equation $y_t = \sum_{i=1}^{p} \phi_i y_{t-i}$,

$$\Rightarrow y_t^h = \sum_{i=1}^{p} \phi_i y_{t-i}^h \quad \text{also} \quad Ay_t^h = \sum_{i=1}^{p} \phi_i (Ay_{t-i}^h) \quad \text{and also} \quad A_1 y_{1t}^h + A_2 y_{2t}^h.$$

$$A_1 y_{1t}^h + A_2 y_{2t}^h = \phi_1 (A_1 y_{1t-1}^h + A_2 y_{2t-1}^h) + \phi_2 (A_1 y_{1t-2}^h + A_2 y_{2t-2}^h) + \cdots + \phi_p (A_1 y_{1t-p}^h + A_2 y_{2t-p}^h),$$

$$\Rightarrow (A_1 y_{1t}^h - \sum_{i=1}^{p} A_1 \phi_i y_{1t-i}^h) + (A_2 y_{2t}^h - \sum_{i=1}^{p} A_2 \phi_i y_{2t-i}^h) \stackrel{?}{=} 0.$$

Since $A_1 y_{1t}^h$ and $A_2 y_{2t}^h$ are separate solution to $y_t = \sum_{i=1}^{p} \phi_i y_{t-i}$ ⇒0+0=0.

$$y_t^p + y_t^h = \sum_{i=1}^{p} \phi_i (y_{t-i}^p + y_{t-i}^h) + x_t,$$

$$\Rightarrow (y_t^p - \sum_{i=1}^{p} \phi_i y_{t-i}^p - x_t) + (y_t^h - \sum_{i=1}^{p} \phi_i y_{t-i}^h) \stackrel{?}{=} 0.$$

Since $y_t^p$ solves the general equation and since $y_t^h$ solves the homogeneous equation ⇒ 0+0=0.

# Homogeneous solutions to DEs with constant coefficients

(1) The 2nd-order systems

$$y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} = 0.$$

Suspect that the homogeneous solutions has the form $y_t^h = A\lambda^t$,

$$A\lambda^t - \phi_1 A\lambda^{t-1} - \phi_2 A\lambda^{t-2} = 0,$$

$$\xrightarrow{\text{divide by } A\lambda^{t-2}} \lambda^2 - \phi_1\lambda - \phi_2 = 0 \leftarrow \text{the characteristic equation},$$

$$\Rightarrow \text{the characteristic roots } \lambda_1, \lambda_2 = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2} \equiv \frac{\phi_1 \pm \sqrt{d}}{2}$$

$$\Rightarrow \text{all homogeneous solutions } y_t^h = A_1\lambda_1^t + A_2\lambda_2^t.$$

$$\begin{cases} y_t^h = A_1\lambda_1^t + A_2\lambda_2^t, & d > 0; \\ y_t^h = A_1\left(\frac{\phi_1}{2}\right)^t + A_2 t\left(\frac{\phi_1}{2}\right)^t, & d = 0;\ \text{where } |\phi_1| > 2 \text{ or } |\phi_1| < 2. \\ y_t^h = A_1 r^t \cos(\theta t + A_2), & d < 0;\ \text{where } r = \sqrt{-\phi_2}\ \cos\theta = \frac{\phi_1}{2\sqrt{-\phi_2}}. \end{cases}$$

# Cont'd: $d > 0$

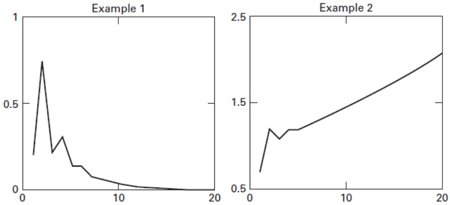


Figure: Example 1: $|\lambda_1|, |\lambda_2| < 1$ vs. Example 2: $|\lambda_1| > 1, |\lambda_2| < 1$

Source: Enders (2015, p.24)

# Cont'd: $d = 0$

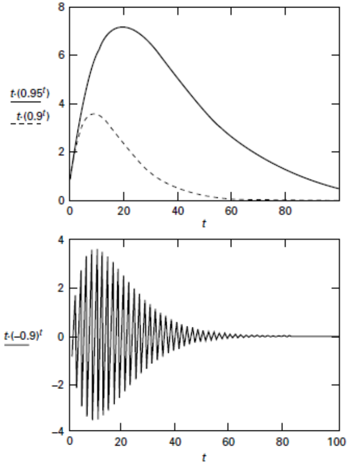


Figure: The homogeneous solution $t\left(\frac{\phi_1}{2}\right)^t$

Source: Enders (2015, p.26)

## Cont'd: $d < 0$

Enders (2015, p. 26-27 & appendix 1.1);

Hamilton (1994, p.14)

# Cont'd: Stability condition

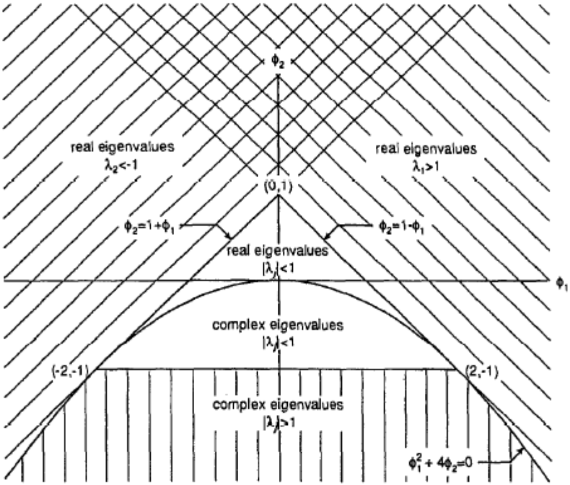


Figure: Summary of dynamics for a 2nd-order DE

Source: Hamilton (1994, p.17)

## Cont'd

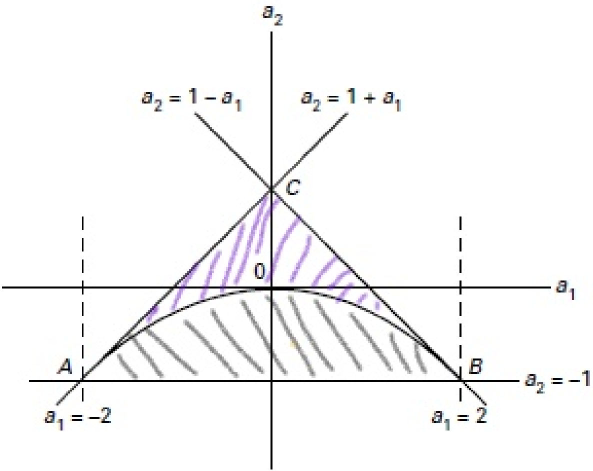


Figure: Characterizing the Stability Conditions

Source: Enders (2015, p.29)

## Cont'd

The general stability conditions can be summarized using triangle ABC in above figure ($a_1 \equiv \phi_1, a_2 \equiv \phi_2$; $d = \phi_1^2 + 4\phi_2 = 0$):

Arc AOB ($d = 0$) is the boundary between cases 1 ($d > 0$) and case 3 ($d < 0$); The region above AOB corresponds to case 1 ($d > 0$);

The region blow AOB corresponds to case 3 ($d < 0$).

In case 1 ($d > 0$ in which the roots are real and distint), stability requires

$$
\begin{aligned}
\lambda_1 = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} < 1, \qquad & \lambda_2 = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2} > -1, \\
\Rightarrow \sqrt{\phi_1^2 + 4\phi_2} < 2 - \phi_1, \qquad & \Rightarrow \phi_1 - \sqrt{\phi_1^2 + 4\phi_2} > -2, \\
\Rightarrow \phi_1^2 + 4\phi_2 < 4 - 4\phi_1 + \phi_1^2, \qquad & \Rightarrow \sqrt{\phi_1^2 + 4\phi_2} < \phi_1 + 2, \\
\Rightarrow \phi_1 + \phi_2 < 1. \qquad & \Rightarrow \phi_2 < 1 + \phi_1.
\end{aligned}
$$

Thus, the region of stability in case 1 ($d > 0$) consists of all points in the region bounded by AOBC;

# Cont'd

In case 2 ($d = 0$ in which the roots are repeated), stability requires

$$|\phi_1| < 2.$$

Thus, the region of stability in case 2 ($d = 0$) consists of all points on arc AOB;

In case 3 ($d < 0$), stability requires

$$r = \sqrt{-\phi_2} < 1 \quad \Rightarrow \quad -\phi_2 < 1 \quad \text{where } \phi_2 < 0.$$

Thus, the region of stability in case 3 ($d < 0$) consists of all points in region AOB.

A succinct way to characterize the stability conditions is to state that the characteristic roots $(\lambda_1, \lambda_2)$ must lie within the unit circle.

If $a_1 \equiv \phi_1 > 0$, the roots $\alpha_1 \equiv \lambda_1 = \frac{\phi_1 + i\sqrt{d}}{2}$ and $\alpha_2 \equiv \lambda_2 = \frac{\phi_1 - i\sqrt{d}}{2}$ can be represented by the two points in the following figure.

# Cont'd

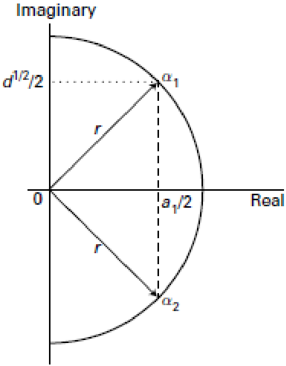


Figure: Characterizing the Stability Conditions

$$r = \sqrt{\left(\frac{\phi_1}{2}\right)^2 + \left(\frac{i\sqrt{d}}{2}\right)^2} = \sqrt{\left(\frac{\phi_1}{2}\right)^2 - \left(\frac{\sqrt{d}}{2}\right)^2} = \sqrt{\frac{\phi_1^2}{4} - \frac{\phi_1^2 + 4\phi_2}{4}} = \sqrt{-\phi_2}.$$

Source: Enders (2015, p.30)

# Cont'd: An example

$$
\begin{aligned}
y_t &= 0.9y_{t-1} - 0.2y_{t-2} + 3, \\
&\Rightarrow y_t - 0.9y_{t-1} + 0.2y_{t-2} = 0, && \text{the homogeneous portion} \\
&\Rightarrow y_{1t}^h = 0.5^t, \quad y_{2t}^h = 0.4^t, && \text{all homogeneous solutions} \\
&\Rightarrow y_t^p = 10, && \text{a particular solution} \\
&\Rightarrow y_t = A_1 0.5^t + A_2 0.4^t + 10, && \text{the general solution} \\
&\rightarrow A_1 = 1 \;\&\; A_2 = 2 \qquad \Leftarrow && y_0 = 13 \;\&\; y_1 = 11.3, \\
&\Rightarrow y_t = 0.5^t + 2 \cdot 0.4^t + 10.
\end{aligned}
$$

We can substitute the solution into the 2nd-order equation to verify that it is correct.

## Cont'd

(2) The pth-order systems $y_t - \sum_{i=1}^{p} \phi_i y_{t-i} = 0$.

Suspect each homogeneous solution to have the form $y_t^h = A\lambda^t$.

To find the value(s) of $\lambda$, we seek the solution for

$$A\lambda^t - \sum_{i=1}^{p} \phi_i A\lambda^{t-i} = 0,$$

$$\xrightarrow{\text{divide by } A\lambda^{t-p}} \lambda^p - \phi_1\lambda^{p-1} - \phi_2\lambda^{p-2} - \cdots - \phi_p = 0,$$

$$y_t^h = A_1\lambda_1^t + A_2\lambda_2^t + \cdots + A_p\lambda_p^t.$$

Stability requires that all real valued $|\lambda_i| < 1$.

Some rules for checking the stability conditions (i.e., all characteristic roots lie inside the unit circle) in higher order systems:

a necessary condition: $\sum_{i=1}^{p} \phi_i < 1$.

a sufficient condition: $\sum_{i=1}^{p} |\phi_i| < 1$.

at least one of $\{\lambda_i\} = 1$ if $\sum_{i=1}^{p} \phi_i = 1 \quad \Rightarrow \quad$ a unit root process.

# Particular solutions to DEs with a variable term

The pth-order system

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t.$$

The appropriate technique depends heavily on the form of the $\{\epsilon_t\}$ process.

Case 1: $\{\epsilon_t\}$ contain only deterministic components.

Case 2: $\{\epsilon_t\}$ contain only stochastic components.

Case 3: $\{\epsilon_t\}$ contain both deterministic and stochastic components.

Refer the reader to Chiang (2005, 4th ed., pp.586-588) and Enders (2015, pp.32-34).

# Cont'd: Case 1

Deterministic processes

(1) $\epsilon_t = 0$

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p}.$$

Note that this is not a homogeneous equation since $c \neq 0$. The trial solution $y_t^p = k$, thus

$$y_t^p = k = \frac{c}{1 - \phi_1 - \phi_2 - \cdots - \phi_p}, \qquad 1 - \phi_1 - \phi_2 - \cdots - \phi_p \neq 0.$$

Try the solution $y_t^p = kt$ when $1 - \phi_1 - \phi_2 - \cdots - \phi_p = 0$,

$$\begin{aligned} kt &= c + \phi_1 k(t-1) + \phi_2 k(t-2) + \cdots + \phi_p k(t-p), \\ (1 - \phi_1 - \phi_2 - \cdots - \phi_p)kt &= c - k(\phi_1 + 2\phi_2 + 3\phi_3 + \cdots + p\phi_p), \\ \Rightarrow 0 &= c - k(\phi_1 + 2\phi_2 + 3\phi_3 + \cdots + p\phi_p), \\ \Rightarrow k &= \frac{c}{\phi_1 + 2\phi_2 + 3\phi_3 + \cdots + p\phi_p}. \end{aligned}$$

In the event that the solution $kt$ fails, sequentially try the solutions $y_t^p = kt^2, kt^3, \ldots, kt^p$.

## Cont'd

(2) $\epsilon_t = ab^{\phi_0 t}$ (in a growth context) where a, b, and $\phi_0$ are constants, e.g.,

$$y_t = c + \phi_1 y_{t-1} + ab^{\phi_0 t}.$$

Try a slotuion of the form $y_t^p = k_0 + k_1 b^{\phi_0 t}$, where $k_0$ and $k_1$ are constants.

$$\begin{aligned}
k_0 + k_1 b^{\phi_0 t} &= c + \phi_1 (k_0 + k_1 b^{\phi_0 (t-1)}) + ab^{\phi_0 t}, \\
k_0 - c - \phi_1 k_0 &= \phi_1 k_1 b^{\phi_0 (t-1)} + ab^{\phi_0 t} - k_1 b^{\phi_0 t}, \\
0 &= 0, \\
\Rightarrow k_0 = \frac{c}{1-\phi_1} \quad &\text{and} \quad k_1 = \frac{ab^{\phi_0 t}}{b^{\phi_0 t} - \phi_1 b^{\phi_0 (t-1)}} = \frac{ab^{\phi_0}}{b^{\phi_0} - \phi_1}, \\
\Rightarrow y_t^p = k_0 + k_1 b^{\phi_0 t} &= \frac{c}{1-\phi_1} + \frac{ab^{\phi_0}}{b^{\phi_0} - \phi_1} b^{\phi_0 t}, \\
\Rightarrow \quad y_t^p \quad \rightarrow \quad k_0 &= \frac{c}{1-\phi_1} \quad \Leftarrow |b^{\phi_0}| < 1.
\end{aligned}$$

Note that we would try another solution $k_0 = kt$ when $\phi_1 = 1$ and $k_1 = at$ when $\phi_1 = b^{\phi_0}$.

# Cont'd

(3) $\epsilon_t = at^{\phi_0}$ (deterministic time trend) where a is a constant and $\phi_0$ is a positive integer. Hence

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + at^{\phi_0}.$$

Try a solution of the form $y_t^p = k_0 + k_1 t + k_2 t^2 + \cdots + k_{\phi_0} t^{\phi_0}$. To find the value of each $k_i$, substitute the particular solution into the above equation. Then select the value of each $c_i$ that results in an identity. Although various values of $\phi_0$ are possible, in economic applications it is common to see models incorporating a linear time trend ($\phi_0 = 1$). Consider an example

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + at.$$

Posit the solution $y_t^p = k_0 + k_1 t$ where $k_0$ and $k_1$ are undetermined coefficients. Substitute to yield ( if $\phi_1 + \phi_2 \neq 1$)

$$k_0 + k_1 t = c + \phi_1[k_0 + k_1(t-1)] + \phi_2[k_0 + k_1(t-2)] + at,$$

$$k_1 t - \phi_1 k_1 t - \phi_2 k_1 t - at = c + \phi_1 k_0 - \phi_1 k_1 + \phi_2 k_0 - 2\phi_2 k_1 - k_0,$$

$$\Rightarrow k_1 = \frac{a}{1-\phi_1-\phi_2} \quad \text{and} \quad k_0 = \frac{c}{1-\phi_1-\phi_2} - \frac{a}{(1-\phi_1-\phi_2)^2}(\phi_1 + 2\phi_2).$$

## Cont'd: Case 2

Finding particular solutions when there are stochastic components in the $\{y_t\}$ process by the method of undetermined coefficients. The key insight of the method is that linear equations have linear solutions.

Hence, the particular solution to a linear DE is necessarily linear. Moreover, the solution can depend only on time, a constant, and the elements of the forcing process $\{\epsilon_t\}$.

To begin, reconsider the 1st-order equation AR(1):

$$y_t = c + \phi_1 y_{t-1} + \epsilon_t.$$

Posit the particular soution (also be named as the challenge solution):

$$y_t^p = k_0 + k_1 t + \sum_{i=0}^{\infty} x_i \epsilon_{t-i},$$

where $k_0$, $k_1$, and all the $x_i$ are the coefficients to be determined.

# Cont'd

Substitute it into the original DE to form

$$
\begin{aligned}
y_t &= c + \phi_1 y_{t-1} + \epsilon_t, \\
\Rightarrow \quad & k_0 + k_1 t + x_0\epsilon_t + x_1\epsilon_{t-1} + x_2\epsilon_{t-2} + \cdots \\
&= c + \phi_1[k_0 + k_1(t-1) + x_0\epsilon_{t-1} + x_1\epsilon_{t-2} + x_2\epsilon_{t-3} + \cdots] + \epsilon_t, \\
\Rightarrow \quad & (k_0 - c - \phi_1 k_0 + \phi_1 k_1) + (k_1 - \phi_1 k_1)t \\
&+(x_0 - 1)\epsilon_t + (x_1 - \phi_1 x_0)\epsilon_{t-1} + (x_2 - \phi_1 x_1)\epsilon_{t-2} + (x_3 - \phi_1 x_2)\epsilon_{t-3} + \cdots = 0,
\end{aligned}
$$

$$
\Rightarrow \begin{cases}
k_0 - c - \phi_1 k_0 + \phi_1 k_1 & = 0 \quad \Rightarrow \quad k_0 = \frac{c}{1-\phi_1}, \\
k_1 - \phi_1 k_1 & = 0 \xRightarrow{\phi_1 \neq 1} k_1 = 0 \Uparrow, \\
x_0 - 1 & = 0 \quad \Rightarrow \quad x_0 = 1, \\
x_1 - \phi_1 x_0 & = 0 \quad \Rightarrow \quad x_1 = \phi_1, \\
x_2 - \phi_1 x_1 & = 0 \quad \Rightarrow \quad x_2 = \phi_1^2, \\
x_3 - \phi_1 x_2 & = 0 \quad \Rightarrow \quad x_2 = \phi_1^3, \\
\vdots & \\
x_i - \phi_1 x_{i-1} & = 0 \quad \Rightarrow \quad x_i = \phi_1^i.
\end{cases}
$$

# Cont'd

For this case, the particular solution is

$$y_t^p = \frac{c}{1-\phi_1} + \sum_{i=0}^{\infty} \phi_1^i \epsilon_{t-i}.$$

Compare this method to the method of iteration

$$\begin{aligned}
y_0 &= c + \phi_1 y_{-1} + \epsilon_0, \qquad\qquad y_1 = c + \phi_1 y_0 + \epsilon_1, \\
y_2 &= c + \phi_1(c + \phi_1 y_0 + \epsilon_1) + \epsilon_2 = c + c\phi_1 + \phi_1^2 y_0 + \phi_1 \epsilon_1 + \epsilon_2, \\
y_t &= c \sum_{i=0}^{t-1} \phi_1^i + \phi_1^t y_0 + \sum_{i=0}^{t-1} \phi_1^i \epsilon_{t-i}, \\
&= c \sum_{i=0}^{t-1} \phi_1^i + \phi_1^t (c + \phi_1 y_{-1} + \epsilon_0) + \sum_{i=0}^{t-1} \phi_1^i \epsilon_{t-i} \\
&= c \sum_{i=0}^{t} \phi_1^i + \phi_1^{t+1} y_{-1} + \sum_{i=0}^{t} \phi_1^i \epsilon_{t-i} \\
&= c \sum_{i=0}^{t+p} \phi_1^i + \phi_1^{t+p+1} y_{-p-1} + \sum_{i=0}^{t+p} \phi_1^i \epsilon_{t-i} \xlongequal{p \to \infty} \frac{c}{1-\phi_1} + \sum_{i=0}^{\infty} \phi_1^i \epsilon_{t-i}.
\end{aligned}$$

## Cont'd

The general solution $y_t = y_t^p + y_t^h$, where $y_t^h = A\phi_1^t$:

$$y_t = \frac{c}{1-\phi_1} + \sum_{i=0}^{\infty} \phi_1^i \epsilon_{t-i} + A\phi_1^t.$$

If we have an initial condition for $y_0$, it follows that

$$y_0 = \frac{c}{1-\phi_1} + \sum_{i=0}^{\infty} \phi_1^i \epsilon_{-i} + A,$$

$$\Rightarrow A = y_0 - \frac{c}{1-\phi_1} + \sum_{i=0}^{\infty} \phi_1^i \epsilon_{-i}.$$

$$\begin{aligned}
\Rightarrow y_t &= \frac{c}{1-\phi_1} + \sum_{i=0}^{\infty} \phi_1^i \epsilon_{t-i} + \left( y_0 - \frac{c}{1-\phi_1} + \sum_{i=0}^{\infty} \phi_1^i \epsilon_{-i} \right) \phi_1^t \\
&= \frac{c}{1-\phi_1} + \left( \sum_{i=0}^{\infty} \phi_1^i \epsilon_{t-i} + \phi_1^t \sum_{i=0}^{\infty} \phi_1^i \epsilon_{-i} \right) + \left( y_0 - \frac{c}{1-\phi_1} \right) \phi_1^t \\
&= \frac{c}{1-\phi_1} + \sum_{i=0}^{t-1} \phi_1^i \epsilon_{t-i} + \left( y_0 - \frac{c}{1-\phi_1} \right) \phi_1^t, \quad \phi_1 \neq 1.
\end{aligned}$$

## Cont'd

Instead if $\phi_1 = 1$ (or $|\phi_1 > 1|$ "nonconcergent sequences", see Enders, 2005, pp. 12, 36), $k_0$ can be any arbitrary constant and $k_1 = c$. The improper form of the particular solution is ($x_i = \phi_1^i$ will explode when $|\phi_1| > 1$)

$$y_t^p = k_0 + k_1 t + \sum_{i=0}^{\infty} x_i \epsilon_{t-i} \quad \rightarrow \quad y_t^p = k_0 + ct + \sum_{i=0}^{\infty} \epsilon_{t-i}.$$

The form of the solution is "improper" because $\sum_{i=0}^{\infty} \epsilon_{t-i}$ may be infinite. Therefore, it's necessary to impose an initial condition ($y_0$ given) to yield

$$y_0 = k_0 + \sum_{i=0}^{\infty} \epsilon_{-i} \quad \Rightarrow \quad k_0 = y_0 - \sum_{i=0}^{\infty} \epsilon_{-i},$$

$$\Rightarrow y_t = \left( y_0 - \sum_{i=0}^{\infty} \epsilon_{-i} \right) + ct + \sum_{i=0}^{\infty} \epsilon_{t-i}$$

$$= y_0 + ct + \left( \sum_{i=0}^{\infty} \epsilon_{t-i} - \sum_{i=0}^{\infty} \epsilon_{-i} \right)$$

$$= y_0 + ct + \sum \epsilon_i, \quad i = 1, 2, \ldots, t.$$

## Cont'd: Another example

AR(1) $\rightarrow$ ARMA(1, 1):

$$Y_t = c + \phi_1 Y_{t-1} + \epsilon_t + \theta_1 \epsilon_{t-1}.$$

Posit the challenge solution and substitute it into the original equation to yield

$$Y_t^p = k_0 + k_1 t + \sum_{i=0}^{\infty} x_i \epsilon_{t-i},$$

$$\Rightarrow k_0 + k_1 t + \sum_{i=0}^{\infty} x_i \epsilon_{t-i} = c + \phi_1 [k_0 + k_1(t-1) + \sum_{i=0}^{\infty} x_i \epsilon_{t-1-i}] + \epsilon_t + \theta_1 \epsilon_{t-1}$$

Matching coefficients of intercept terms, coefficients of terms containing t, and coefficients on all terms containing $\epsilon_t, \epsilon_{t-1}, \epsilon_{t-2}, \ldots$ yields $\cdots$

If $\phi_1 \neq 1$,

If $\phi_1 = 1$,

# Cont'd

AR(1) $\rightarrow$ ARMA(1, 1) $\rightarrow$ AR(2)

$$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \epsilon_t.$$

Since it's a 2nd-order equation, we use the challenge solution

$$Y_t = k_0 + k_1 t + k_2 t^2 + \sum_{i=0}^{\infty} x_i \epsilon_{t-i},$$

where $k_0, k_1, k_2$, and the $x_i$ are the undetermined coefficients.

Substituting it into the original equation yields $\cdots$

We will continue to talk about ARMA(1, 1) and AR(2) in Lecture 2.

## Cont'd: Case 3

See some examples using dynare codes.

1. Introduction (Learning Motivation, etc.)
2. Syllabus
3. DEs with Constant Coefficients and Constant Terms
4. Blackboard-Writing of Solving DEs
5. Cite a Model As an Example
6. DEs with Constant Coefficients and Variable Terms
7. **Topics for Next Week**

# Contents

Stay at Lecture 1,
but turn to Hamilton (1994, ch.1-2).

Chapter 2 Lag Operators;

Chapter 1 Matrix Operations.