Lecture Notes 1: Solow Growth Model

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Solow model (Solow, 1959) is the starting point of most of the dynamic macroeconomic theories. It introduces dynamics and transitions into the aggregate economy. We first list several key assumptions regarding the model's setup. To be consistent with future lectures, we assume that the time is discrete.

1 Model Setup

1. **Production Technology**

   The firm produces final goods by combining capital K_t and labor L_t. The production function is assumed to take the form

   $$Y_t = K_t^{\alpha} (A_t L_t)^{1-\alpha}, \tag{1}$$

   where A_t is the neutral technology process, $\alpha \in (0, 1)$ is the capital share. The production function is concave in K and L: $\frac{\partial^2 Y_t}{\partial K_t^2} < 0$ and $\frac{\partial^2 Y_t}{\partial L_t^2} < 0$. This assumption implies that the marginal product of capital (MPK) is decreasing in K and the marginal product of labor (MPL) is decreasing in L. The concavity of production function ensures the uniqueness of balance growth path and the global convergency.

2. **Labor Supply**

   The labor input L_t equals the population (*full employment*) and grows at a constant rate $n > 0$, that is

   $$L_t = (1 + n) L_{t-1}. \tag{2}$$

3. **Technology Progress**

   The neutral technology A_t grows at a constant rate $g > 0$:

   $$A_t = (1 + g) A_{t-1}. \tag{3}$$

   Note that Solow model introduces exogenous growth by assuming the population and the technology are growing over time.

4. **Consumption and Saving**

   Households are representative. The saving, S_t, and consumption, C_t, decisions are static:

$$S_t = sY_t, \tag{4}$$

$$C_t = (1 - s) Y_t, \tag{5}$$

   where s is the constant saving rate. Note that the above decisions may not necessarily optimal for the household. Hence, the Solow model does not consider the micro-level optimization decisions. This assumption will be relaxed in the Ramsey model in future lectures.

5. **Capital Accumulation**

   To make the model be dynamic, we need to introduce the dynamic process of capital accumulation:

$$K_{t+1} = (1 - \delta) K_t + I_t, \tag{6}$$

   where K_t is the beginning-of-period capital stock. That is, K_t is predetermined at period $t-1$, therefore K_t is a state variable.

6. **Capital Market Clearing**

$$I_t = S_t. \tag{7}$$

   In the closed economy, aggregate investment is always equal to aggregate savings. However, in the open economy, two variables may not necessarily be the same. The difference between saving and investment is the current account balance.

2 Dynamics

2.1 Dynamic System

The full system is given by equations (1) to (7). From (4) to (7), we can simplify the system into one difference equation

$$K_{t+1} = (1 - \delta) K_t + s (K_t)^\alpha (A_t L_t)^{1-\alpha}. \tag{8}$$

To remove the non-stationary trend in the last equation, we need to detrend each variable. Divide the both sides of (8) by $A_t L_t$, and define new variable $x_t \equiv \frac{X_t}{A_t L_t}$. We can rewrite (8) as

$$k_{t+1} (1 + g) (1 + n) = (1 - \delta) k_t + sk_t^\alpha. \tag{9}$$

Rearranging the terms yields

$$(1+g)(1+n)(k_{t+1}-k_t) = sf(k_t) - (\delta + g + n + g \times n) k_t. \tag{10}$$

where $f(k) = k^\alpha$. For simplicity, we ignore the terms with the coefficient $g \times n$. We then obtain

$$(1+g+n)(k_{t+1}-k_t) = sf(k_t) - (\delta + g + n) k_t. \tag{11}$$

The above difference equation determines the process of the effective capital stock k_t $\left(\text{or } \frac{K_t}{A_t L_t}\right)$.

Before we discuss the dynamics implied by (11), we first look at the stationary growth in the long run, where $\Delta k_t = 0$ or $k_{t+1} = k_t$.

2.2 Balanced growth path (BGP):

BGP is the path on which all the variables of the model grow at a constant rate $n+g$, and the detrended variables (e.g. k_t) stay constant. On the balance growth path, k_t is constant: $k_t = k^*$, for all t. It can be shown that under the Cobb-Douglas technology, there exists a unique k^* to solve the equation

$$s(k^*)^\alpha - (\delta + g + n) k^* = 0, \tag{12}$$

or

$$k^* = \left(\frac{s}{\delta + g + n}\right)^{\frac{1}{1-\alpha}}. \tag{13}$$

On the balance growth path, the capital stock, K_t, is given by

$$K_t^B = k^* A_t L_t. \tag{14}$$

Plugging K_t^B into (1) and (5), balance growth output, consumption and saving, are given by

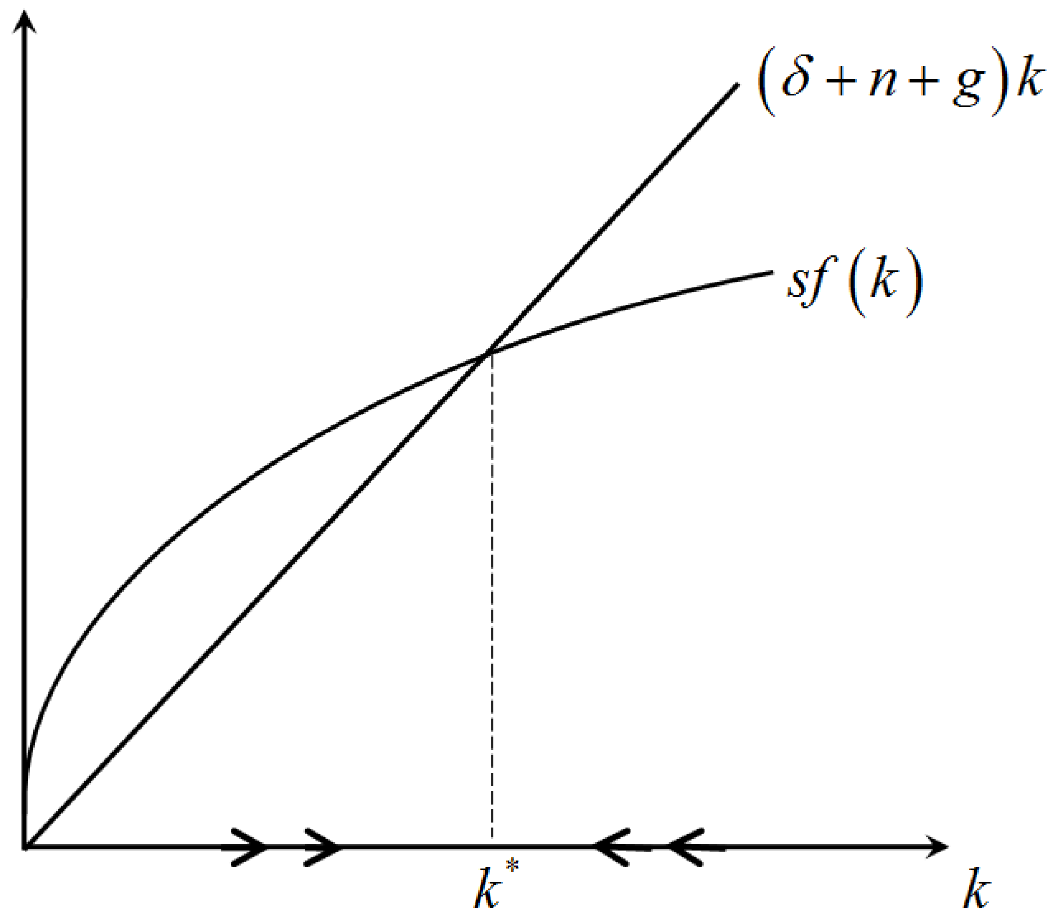
$$Y_t^B = (k^*)^\alpha A_t L_t, \tag{15}$$

$$S_t^B = s(k^*)^\alpha A_t L_t, \tag{16}$$

$$C_t^B = (1-s)(k^*)^\alpha A_t L_t. \tag{17}$$

The BGP indicates that long-run growth does not depend on the initial state and capital accumulation. The long-run growth rate is only determined by the growth of technology g and population n.

Figure 1: Phase diagram for the dynamics of k_t



2.3 Dynamics of k_t and y_t

The concavity of production technology ensures the global convergency of k_t. To proof this, suppose $k_t < k^*$, the right hand side of (11) implies $k_{t+1} - k_t > 0$. Thus k_t will monotonically increase until $k_t = k^*$. For the case of $k_t > k^*$, similarly we have $k_{t+1} < k_t$, and k_t will monotonically decrease until $k_t = k^*$. Therefore, for any initial value of k_t, it will eventually converge to the steady-state k^*. The phase diagram below illustrates the dynamics of k_t.

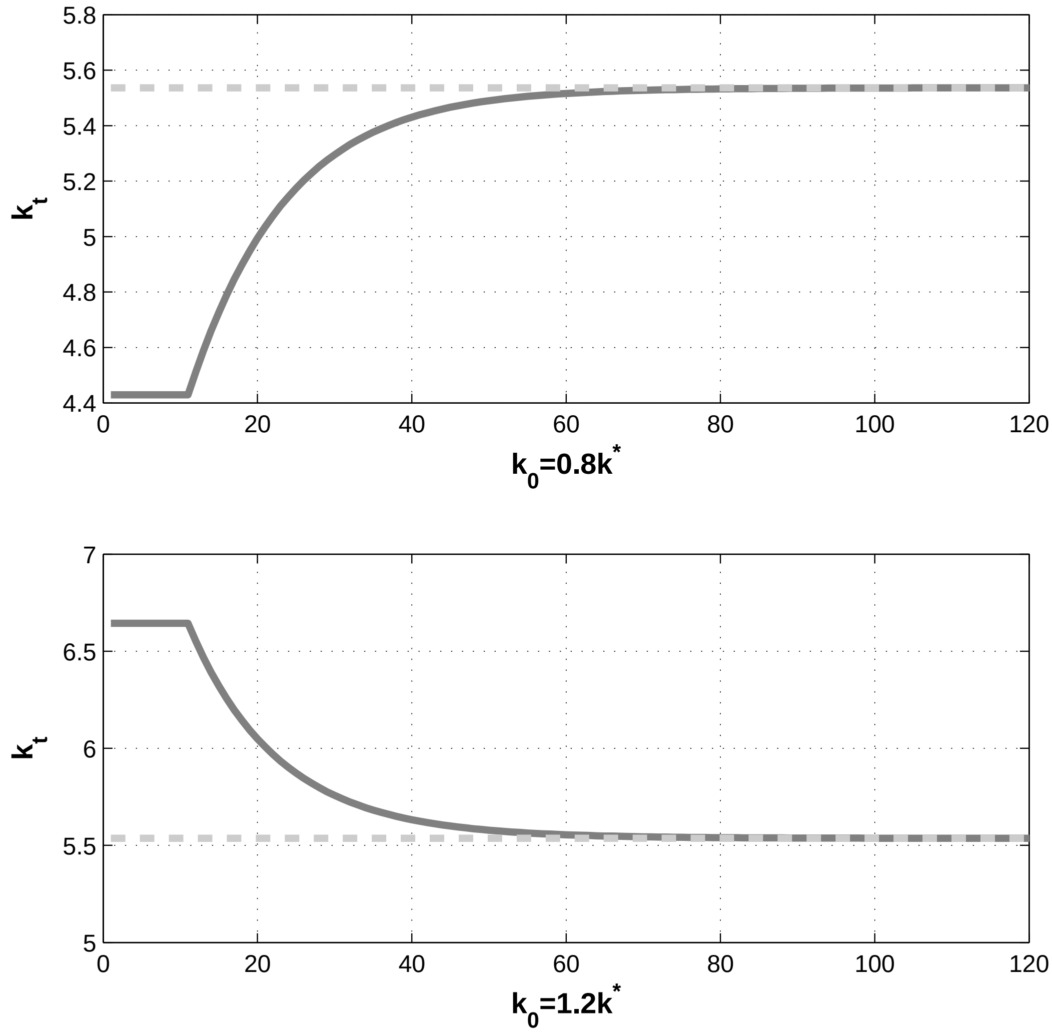
To exactly see how the capital k_t transits from the initial state to the steady-state k^*, we do the following simulation. First, we set the values of deep parameters as

$$s = 0.4, n = 2\%, g = 5\%, \alpha = 0.5, \delta = 0.1.$$

We consider two scenarios: (1) $k_0 = 0.8k^*$; (2) $k_0 = 1.2k^*$. Figure 2 shows that capital k_t monotonically converges to its steady state k^*.

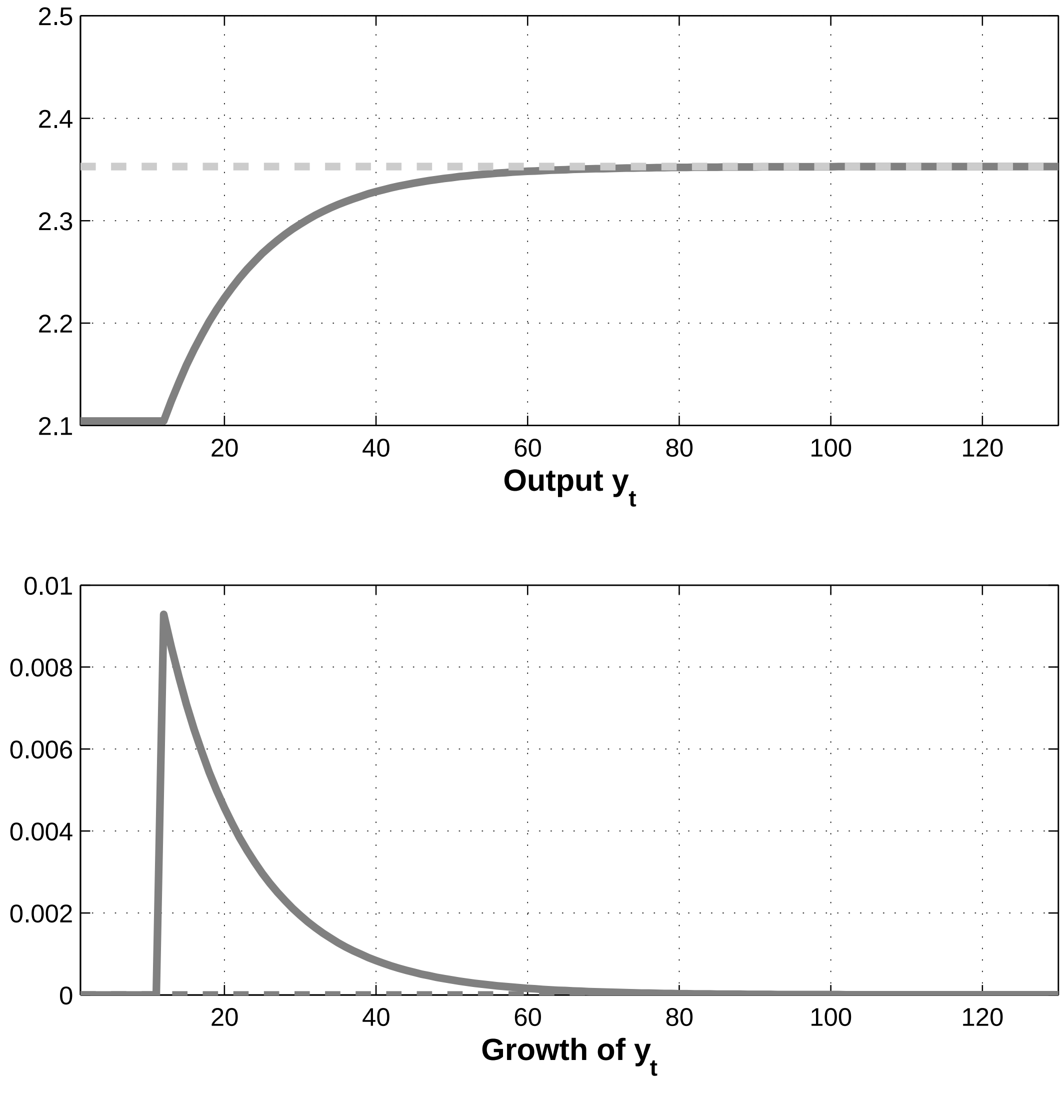
Suppose that poor country and rich country have the same economic structure (same parameter values), Figure 2 shows that the poor country (low initial value) grows fast at the beginning,

Figure 2: Transition paths of capital for different initial values ($k_0 = 0.8k^*, 1.2k^*$)



Notes: This figure plots the transition dynamics for the capital stock k_t. The vertical axis is the level of capital k_t. The horizontal axis is the time period. The dashed line indicates the steady-state level of capital k^*.

Figure 3: Transition path of output ($k_0 = 0.8k^*$)



Notes: This figure plots the transition dynamics for the output y_t when the initial capital is $k_0 = 0.8k^*$. The vertical axis is the output level y_t or growth rate $\Delta \log(y_t)$. The horizontal axis is the time period. The dashed line indicates the steady state.

afterward the growth rate declines. Eventually, the poor country catches up with the rich country and converges to a unique steady state. The aggregate output (non-detrended) Y_t will be on the balanced growth path.

2.4 The effect of saving rate (s) on the long-run growth

To see how the saving rate affects the steady state k^*, we take logs of both sides of (13) and take derivative w.r.t s,

$$\frac{\partial \log k^*}{\partial \log s} = \frac{1}{1-\alpha}. \tag{18}$$

For the steady-state output, we have

$$\frac{\partial \log y^*}{\partial \log s} = \frac{\alpha}{1-\alpha}. \tag{19}$$

A higher capital share implies a larger effect of saving rate on output. Since the saving rate has a positive effect on capital k^*, a rise in capital intensity in the economy implies that raising the saving rate will increase the output more.

As the consumption is given by $c^* = (1-s)(k^*)^{\alpha}$, we further have

$$\frac{\partial \log c^*}{\partial \log s} = -\frac{s}{1-s} + \frac{\alpha}{1-\alpha}. \tag{20}$$

Golden Rule: the consumption c^* achieves the optimal level, which implies that $\frac{\partial \log c^*}{\partial \log s} = 0$ or $s = \alpha$.

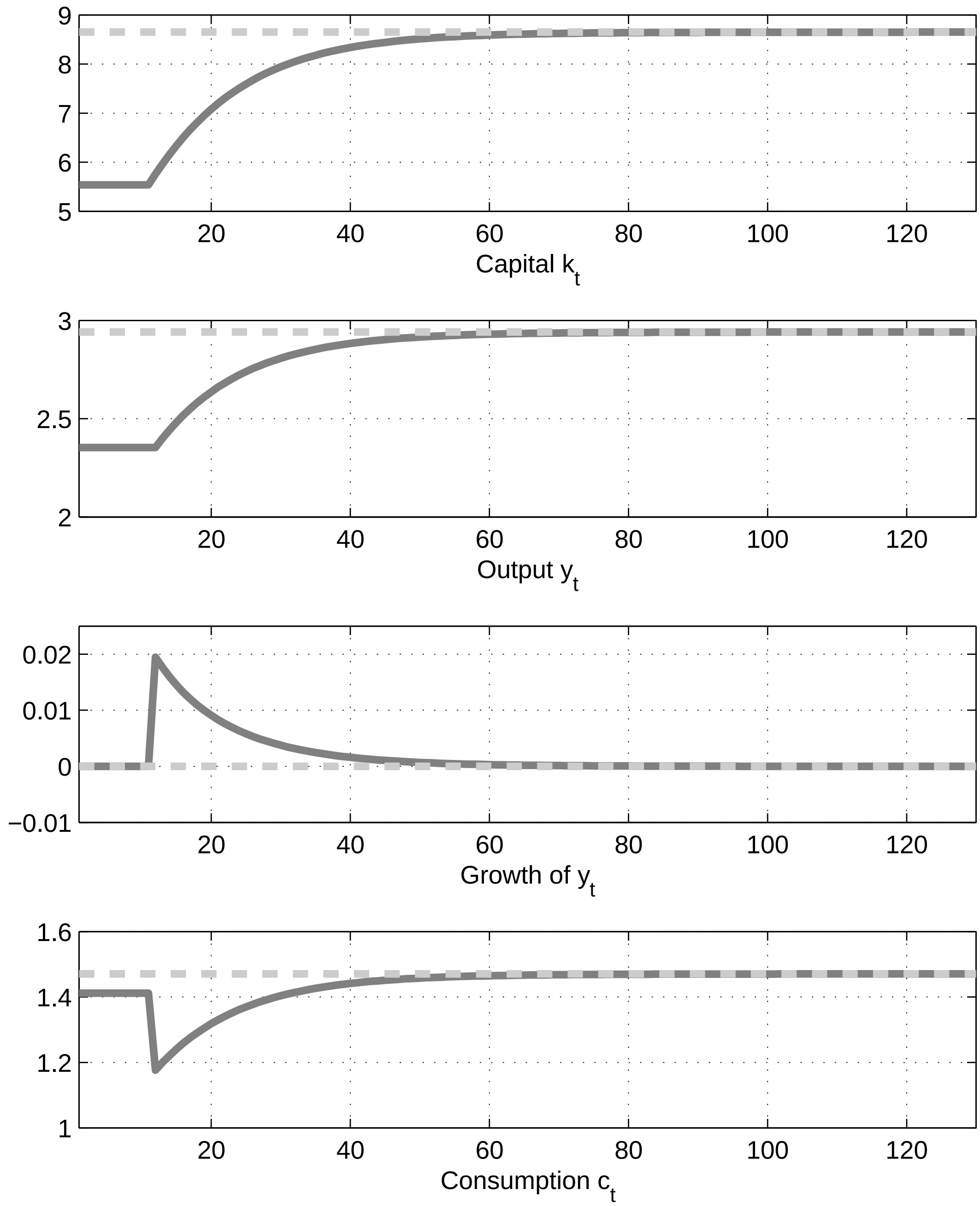
To give a numerical example, Figure 4 computes the dynamic path for the transition from the old steady state ($s = 0.4$) to the new steady state with high saving rate ($s = 0.5$).

2.5 Short-run dynamics

The above discussions are related to the long-run dynamics. I.e., we study the dynamic path of the economy transits from the initial state to the steady state. Now, we discuss the short-run dynamics around the steady state. We first linearize the equation (11) around the steady state k^* :

$$\begin{aligned}(k_{t+1}-k^*)-(k_t-k^*) &= \frac{\alpha s (k^*)^{\alpha-1}-(\delta+g+n)}{1+g+n}(k_t-k^*) \\ &= -\frac{(1-\alpha)(\delta+g+n)}{1+g+n}(k_t-k^*). \end{aligned} \tag{21}$$

Figure 4: Transition path to the steady state of high saving rate



Notes: This figure plots the transition dynamics for the economy when the saving rate increases from 0.4 to 0.5. The vertical axis is for the real variables. The horizontal axis is the time period. The dashed line indicates the new steady state.

The second line is due to the relationship $s\left(k^*\right)^\alpha = \left(\delta+g+n\right)k^*$. Define the convergent speed λ as $\frac{(1-\alpha)(\delta+g+n)}{1+g+n}$. Equation (21) implies

$$k_t - k^* = \left(k_0 - k^*\right)\left(1-\lambda\right)^t. \tag{22}$$

Last equation describes the evolution of capital around the steady state. Regarding the output, similarly we have

$$y_t - y^* = \left(y_0 - y^*\right)\left(1-\lambda\right)^t. \tag{23}$$

Given our previous calibration, λ is equal to 0.08, implying that it takes approximate 8 years ($\simeq \frac{\ln 0.5}{\ln(1-\lambda)}$) to get the halfway of the balance-growth-path value.

2.6 Further Discussion

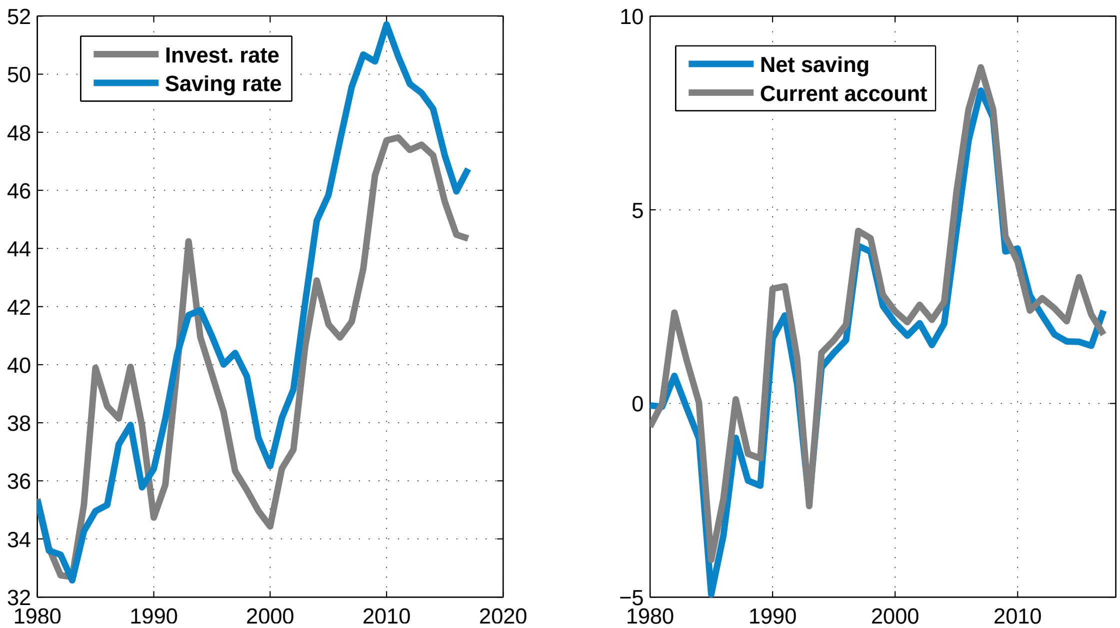
The Solow model is a good framework to understand economic growth. One natural question is: can we directly apply the Solow model for understanding the Chinese Economy? To answer this question, we need to think more deeply about the crucial assumptions made in the model. One important prediction it makes is that a high saving rate may induce a high level of income in the steady-state because one country can accumulate a high level of capital, see equation (13). This argument can be applied to advanced countries such as the United States, but may not be true for developing countries like China. The key assumption in the model is that there are no market frictions. In particular, the model assumes a perfect financial market, which is reflected by two features in the model: (i) the total savings in the household side can be sufficiently channeled to firm's investment, i.e., $S_t = I_t$; (ii) firms invest all of their capital to production purpose, i.e., I_t fully contributes to the production capital without the efficiency loss. However, if one country has less developed financial market, i.e., the total savings cannot be efficiently allocated to the production sector, i.e., the feature (i) fails, and the non-financial firms do not invest in production capital, i.e., the feature (ii) fails, then we cannot conclude that a high saving rate is good for one country's economic growth.

For the Chinese economy, the financial market is underdeveloped. The savings cannot be efficiently allocated to the domestic production side (credit misallocation), resulting in a severe credit shortage for small and private business, and a large imbalance in the financial account and current account (huge foreign reserve and trade surplus).

Figure 5 plots the saving rate (defined as the gross domestic saving as percentage of GDP) and the investment rate (defined as the gross domestic capital formation as percentage of GDP). It shows that the saving rate in China is constantly above the investment rate. The gap of these two series reflects the capital outflow (lending to foreign countries) or current account surplus.

Also the non-financial firms invest a lot in the nonproduction (or financial) assets, for instance,

Figure 5: Saving and investment rate in China



real estate assets. These investment behaviors may crowd out resources allocated to production sector and dampen the GDP growth. Figure 6 shows that an upward trend in the share of financial assets of the firm's total assets (the solid line) is associated with a housing boom (the dashed line). The figure reveals that the strong demand for investing in real estate markets from the firm side may largely contribute to the surge in house prices. The data also imply that a higher return on real estate investment may boost the firm's financial investment and lead to an expansion in the housing sector. The correlation between the share of financial assets and real GDP is -0.43, and the correlation between the share of financial assets and real house prices is 0.41.

3 Extensions

3.1 Natural Resources and Land

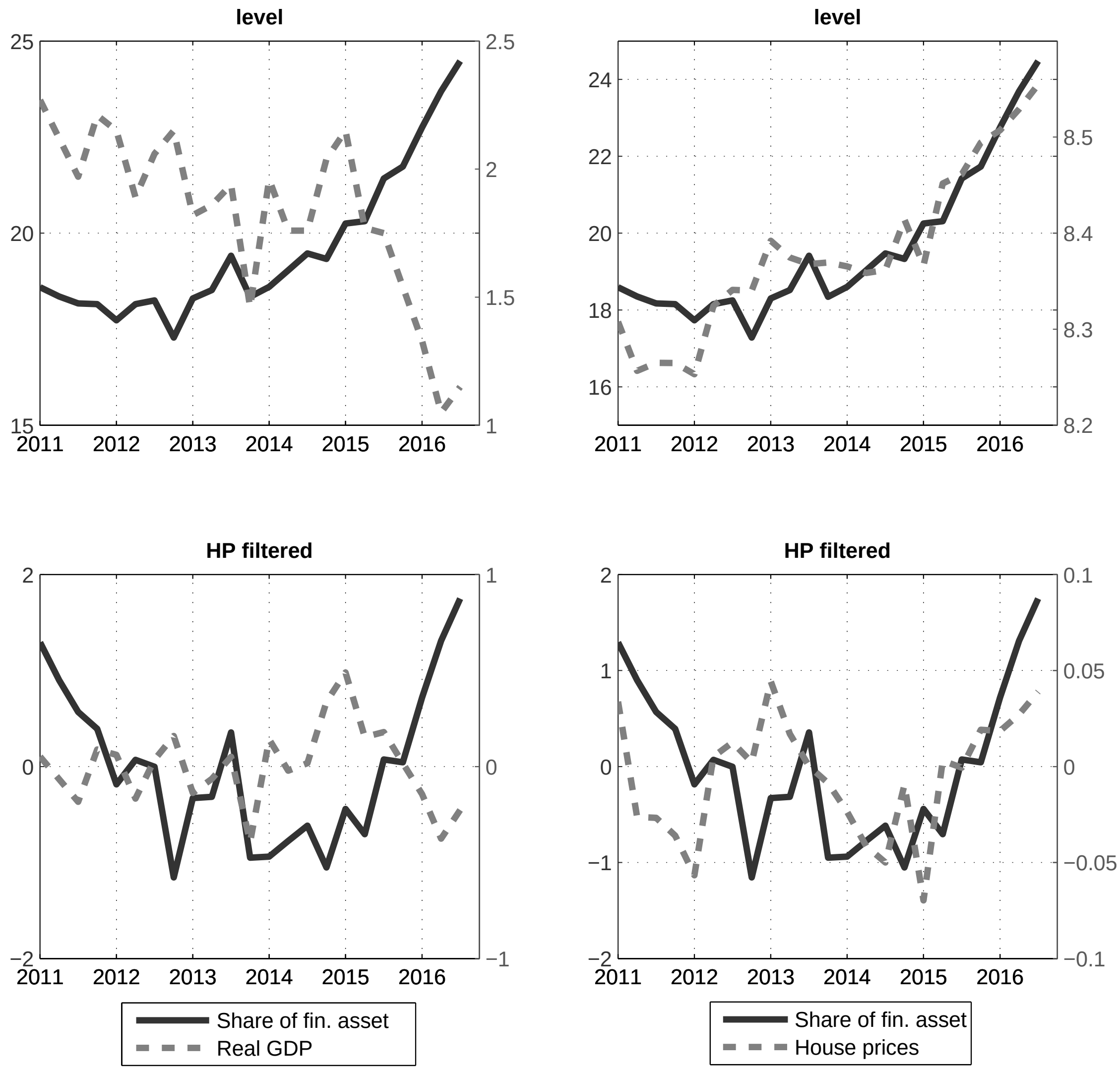
We extend the benchmark Solow model by augmenting production function with two additional inputs: natural resources R_t and land T_t. Specifically, we assume that the production function is

$$Y_t = K_t^{\alpha} R_t^{\beta} T_t^{\gamma} \left(A_t L_t\right)^{1-\alpha-\beta-\gamma}, \tag{24}$$

where $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\alpha + \beta + \gamma < 1$. The process of natural resource R_t is assumed to follow

$$R_t = \left(1 - b\right) R_{t-1}, \tag{25}$$

Figure 6: Share of financial assets in firm side and aggregate economy in China



and the land is assumed to be constant, i.e., $\Delta T_t = 0$. On the balance growth path, it can be shown that capital and output have the same growth rate. This is because (8) implies

$$\frac{K_{t+1}}{K_t} = (1-\delta) + s\frac{Y_t}{K_t}. \tag{26}$$

On BGP, Y_t/K_t is constant or the growth rates satisfy $g_Y = g_K$.

To compute the growth rate of output on BGP, we take logs of both sides of (24) and get

$$g_Y = \alpha g_Y - \beta b + (1-\alpha-\beta-\gamma)(g+n), \tag{27}$$

or

$$g_Y = \frac{-\beta}{1-\alpha}b + \frac{1-\alpha-\beta-\gamma}{1-\alpha}(g+n). \tag{28}$$

Define $\tilde{g}_Y = g + n$ the growth rate where capital and labor are the only inputs. Since $\frac{1-\alpha-\beta-\gamma}{1-\alpha} < 1$ and $-\frac{\beta}{1-\alpha} b < 0$, the growth rate of output in the model with natural resource g_Y is less than that in baseline model $\tilde{g}_Y$. That is, the natural resources (limited supply) reduce the BGP growth rate. The gap between g_Y and $\tilde{g}_Y$ is called the "growth drag".

3.2 Growth Trap: Production with Fixed Cost

So far there is no friction in the Solow model. We now introduce one friction—fixed production cost. You may see that with this minor extension, the dynamics of the model will change a lot. Suppose that each period, in order to make production the producer has to pay a fixed amount of cost ξ. In reality, this cost may be due to the rental rate of land, loan payment, etc. The production function for y_t ($Y_t/A_t/L_t$) takes the form of

$$y_t = \max\left\{k_t^{\alpha} - \xi, 0\right\}. \tag{29}$$

The difference equation (11) now becomes

$$(1 + g + n)(k_{t+1} - k_t) = s \max\left\{k_t^{\alpha} - \xi, 0\right\} - (\delta + g + n) k_t. \tag{30}$$

It can be shown that there exist two k such that

$$s \max\left\{k^{\alpha} - \xi, 0\right\} = (\delta + g + n) k. \tag{31}$$

Denote them as k^* and k^{**}, where $k^* < k^{**}$. As shown in Figure 7, k^{**} is stable in the sense that for any capital $k_t > k^*$, k_t would eventually converge to k^{**}. While k^* is not stable because for any $k_t < k^*$, k_t would converge to the zero point.

3.3 Application: Middle Income Trap

The middle income trap is an economic development situation, where a country which attains a certain income (due to given advantages) will get stuck at that level. Solow model with fixed production cost may capture the main idea. Consider two dynamic systems

$$(1 + g + n)(k_{t+1} - k_t) = s z^l k_t^{\alpha} - (\delta + g + n) k_t, \tag{32}$$

$$(1 + g + n)(k_{t+1} - k_t) = s \max\left\{z^h k_t^{\alpha} - \xi, 0\right\} - (\delta + g + n) k_t, \tag{33}$$

where $z^l < z^h$. The economy starts with a low level K will trap at the K^{mid} equilibrium. To pull out the economy from middle income trap, the government need to either reduce the friction ξ or raise the aggregate capital K. Figure 8 provides a graphical illustration.

Figure 7: Growth Trap

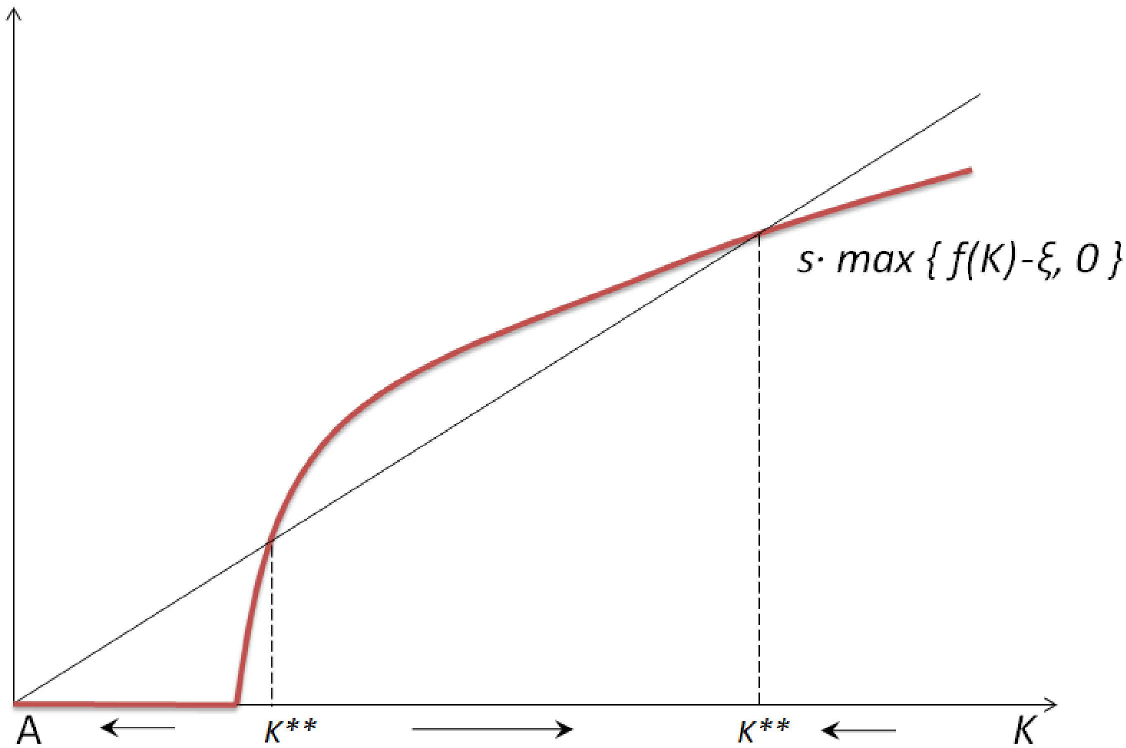


Figure 8: Middle Income Trap