Lecture Notes 2: Neoclassical Growth Theory: Ramsey-Cass-Koopmans Model

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The Solow model does not consider individual optimal decisions. The model's dynamic structure is simply introduced by capital accumulation rule. The consumption rule is exogenously given, as a result, the saving rate is exogenous and constant. In this lecture, we will investigate the Ramsey-Cass-Koopmans model, in which the micro-level optimal behaviors are seriously modeled. In particular, the saving rate is endogenously determined by the household optimization decisions.

1 Model Setup

The economy has representative firms and households The market is competitive. There is only one good, it can be used either for consumption or investment. The production technology is given by

$$Y_t = K_t^{\alpha} \left(A_t L_t\right)^{1-\alpha}, \tag{1}$$

where L_t is the total labor input, K_t is physical capital, A_t is the exogenous technology, growing at rate $g > 0$,

$$A_t = \left(1 + g\right) A_{t-1}. \tag{2}$$

1.1 Firm's Decision

Each period, from the market the representative firm hires labor L_t with the wage rate w_t, and rents capital K_t with the rental rate r_t. The firm chooses L_t and K_t to maximize its profit

$$\Pi_t = Y_t - r_t K_t - w_t L_t. \tag{3}$$

The demand of capital and labor are given by following optimal conditions

$$r_t = \frac{\partial F\left(K_t, A_t L_t\right)}{\partial K_t} = \alpha K_t^{\alpha-1} \left(A_t L_t\right)^{1-\alpha}, \tag{4}$$

$$w_t = \frac{\partial F(K_t, A_t L_t)}{\partial L_t} = (1-\alpha) A_t K_t^{\alpha} (A_t L_t)^{-\alpha}. \tag{5}$$

As the production function is constant return to scale, we have

$$\frac{\partial F(K_t, A_t L_t)}{\partial K_t} K_t + \frac{\partial F(K_t, A_t L_t)}{\partial L_t} L_t = Y_t. \tag{6}$$

Plugging factor demands (4) and (5) into last equation gives us

$$Y_t = r_t K_t + w_t L_t, \tag{7}$$

or $\Pi_t = 0$. That is, the constant return to scale of production function implies zero profit.

1.2 Household's Decision

The representative household is a big family, in which there are N_t members. Each individual inelastically supplies one unit labor. Therefore, the total labor supply is N_t. The population N_t is assumed to grow at the rate n

$$N_t = (1+n) N_{t-1}. \tag{8}$$

The representative household maximizes following life-time utility

$$U = \sum_{t=0}^{\infty} \rho^t [N_t u(C_t)], \tag{9}$$

where C_t is the consumption of a family member, $u(C_t)$ is the corresponding utility level, and ρ is the discount rate for the future. In particular, the utility function is assumed to be $u(C) = \log c_t$.

The budget constraint for the household is given by

$$C_t N_t + K_{t+1} - (1-\delta) K_t = r_t K_t + w_t N_t. \tag{10}$$

The optimization problem of representative household is to maximize (9) subject to (10).

1.3 Competitive Equilibrium

In the competitive equilibrium, the household and the firm achieve their individual optimum, and each market clears. In particular, $N_t = L_t$. The budget constraint (10) and the input demands (4) and (5) jointly imply the resource constraint is

$$C_t L_t + K_{t+1} - (1-\delta) K_t = Y_t. \tag{11}$$

1.4 Dynamic System

Since there are two potential growth trends, to solve the model we need to first transform the economy to a stationary one. Define the detrended variables as $x_t \equiv \frac{X_t}{A_t L_t}$, while for the consumption of each individual, we define $c_t = \frac{C_t}{A_t}$. The life-time utility can be rewritten as

$$\begin{aligned} U &= \sum_{t=0}^{\infty} \rho^t \left[N_t \frac{c_t^{1-\theta} A_t^{1-\theta} - 1}{1-\theta} \right] \\ &= \sum_{t=0}^{\infty} \rho^t \left[N_t A_t^{1-\theta} \frac{c_t^{1-\theta}}{1-\theta} \right] \\ &= N_0 \sum_{t=0}^{\infty} \beta^t \log c_t, \end{aligned} \tag{12}$$

where $\beta = (1+n)\rho$. To guarantee that the utility function is not explosive, we need to assume $\beta < 1$.[1] Without loss of generality, we set $A_0 = N_0 = 1$. The budget constraint can be rewritten as

$$c_t + k_{t+1}(1+n)(1+g) - (1-\delta) k_t = f(k_t), \tag{13}$$

where $f(k) = k_t^{\alpha}$. The competitive equilibrium is equivalent to the solution of following social planner's optimization problem[2]

$$\max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \log c_t, \tag{14}$$

subject to (13). Notice that in the above optimization problem, the capital stock k_t is determined in period $t-1$, therefore k_t is a predetermined variable (state variable) in period t. That is, in each period t, the social planner takes k_t as given and make optimal decisions for c_t and k_{t+1}.

Let λ_t denote the Lagrangian multiplier of (13). The social planner's problem can be rewritten as

$$\mathcal{L} = \max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \left\{ \log c_t + \lambda_t \left[f(k_t) - c_t - k_{t+1}(1+n)(1+g) + (1-\delta) k_t \right] \right\}. \tag{15}$$

The first order condition for consumption c_t is obtained by the condition $\frac{\partial \mathcal{L}}{\partial c_t} = 0$, or equivalently

$$1/c_t = \lambda_t. \tag{16}$$

[1] More rigoriously, to guarantee U is finite, we need the growth rate of $\beta^t \frac{c_t^{1-\theta}}{1-\theta}$ less than zero, or equivalently $(1-\theta)\frac{c_t - c_{t-1}}{c_{t-1}} + \log \beta < 0$. As in the steady state, $\frac{c_t - c_{t-1}}{c_{t-1}} = 0$, we have $\beta < 1$.

[2] According to the first fundamental theorem of welfare economics, if there are no market frictions, the competitive equilibrium in the decentralized economy is equivalent to the optimal allocation derived from a social planner's optimization problem.

The above condition indicates that the Lagrangian multiplier λ_t (or the shadow price) essentially measures the marginal utility in period t.

The first order condition for capital stock k_{t+1} is obtained by the condition $\frac{\partial \mathcal{L}}{\partial k_{t+1}} = 0$, or equivalently

$$\lambda_t = \frac{\beta}{(1+n)(1+g)} \lambda_{t+1} \left[f'(k_{t+1}) + (1-\delta) \right]. \tag{17}$$

The LHS in the last equation indicates the cost of increasing one unit of capital stock in next period. Accumulating one more capital sacrifices one unit consumption in current period, which derives λ_t units of utility. The RHS in the last equation measures the marginal benefit for one more capital stock in next period. An extra unit of capital in the next period will produce $f'(k_{t+1})$ units of output and remains $1-\delta$ unit after the depreciation. The marginal return $f'(k_{t+1}) + (1-\delta)$ can be translated to the utility by multiplying λ_{t+1}. Since the marginal return happens in period $t+1$, the discount factor $\frac{\beta}{(1+n)(1+g)}$ needs to be considered.

We have a full dynamic system, which consists of three equations: one resource constraint (13) and two optimal conditions (16) and (17), to determines three unknowns $\{c_t, k_{t+1}, \lambda_t\}$.

Combining (16) and (17) yields

$$\frac{\Delta c_{t+1}}{c_t} = \frac{\beta}{(1+n)(1+g)} \left[f'(k_{t+1}) + (1-\delta) \right] - 1. \tag{18}$$

We can further write the resource constraint (13) as

$$\Delta k_{t+1} = \frac{1}{(1+n)(1+g)} \left\{ f(k_t) - \left[(1+n)(1+g) - (1-\delta) \right] k_t - c_t \right\}. \tag{19}$$

The dynamic system is fully described by the difference equation system (18) and (19). Note that given any initial state (c_0, k_0), the system (18) and (19) only provide necessary conditions for the optimal paths of $\{c_t, k_t\}$.

1.5 The Steady State and Phase Diagram

Before the discussion of dynamics, we first solve the steady state, where c_t and k_t are constant over time, i.e., $\Delta c_t = \Delta k_t = 0$.

According to (18), when $\Delta c_t = 0$, the steady-state capital k^* satisfies following equation

$$f'(k) = \frac{(1+n)(1+g)}{\beta} - (1-\delta). \tag{20}$$

Remember that $f(k) = k^{\alpha}$, we can further solve the steady state capital k^* as

$$k^* = \left\{ \frac{1}{\alpha} \left[\frac{(1+n)(1+g)}{\beta} - (1-\delta) \right] \right\}^{\frac{1}{\alpha-1}}. \tag{21}$$

The above result indicates that the capital stock, in the long run, is determined by the growth rates of population and technology. This prediction is consistent with that in the Solow model.

Since $f(k) = k^{\alpha}$ is a concave function. Therefore, $f'(k)$ is decreasing in k. If $k_t > k^*$, we have $\frac{\beta}{(1+n)(1+g)} [f'(k_t) + (1-\delta)] - 1 < 0$, or $\Delta c_t < 0$. If $k_t < k^*$, we have $\Delta c_t > 0$.

According to (19), when $\Delta k_{t+1} = 0$, the steady-state c^* and k^* satisfy following equation

$$c = f(k) - [(1+n)(1+g) - (1-\delta)] k. \tag{22}$$

which describes a hump-shaped curve in the (c, k) space. For those (c, k) above the curve, we have $\Delta k_t < 0$. And for those (c, k) below the curve, we have $\Delta k_t > 0$.

Note that (20) is the set of all combinations of (c, k) that ensure $\Delta c_t = 0$, and the (22) is the set of all (c, k) that ensure $\Delta k_t = 0$. We call these two curves as the equilibrium locus. The intersection of them is the steady state.

The phase diagram consists of four areas separated by the locus (20) and the locus (22).

Area I: $c_t \uparrow$ and $k_t \downarrow$. Any $c - k$ pair in this area indicates that capital stock is relatively high and consumption is relatively low, thus according to (18) and (19), consumption will increase and capital will decrease. The above dynamics imply that if the system starts from any point in this area, consumption and capital will eventually *diverge* from the steady-state point (c^*, k^*).

Area II: $c_t \downarrow$ and $k_t \downarrow$. Any $c - k$ pair in this area indicates that both capital stock and consumption are relatively high, thus according to (18) and (19), both of them will monotonically decrease as long as (c_t, k_t) stays in this area. The above dynamics imply that in this area there exist *some* $c - k$ pairs as initial points, from which consumption and capital will eventually *converge* to the steady-state point (c^*, k^*).

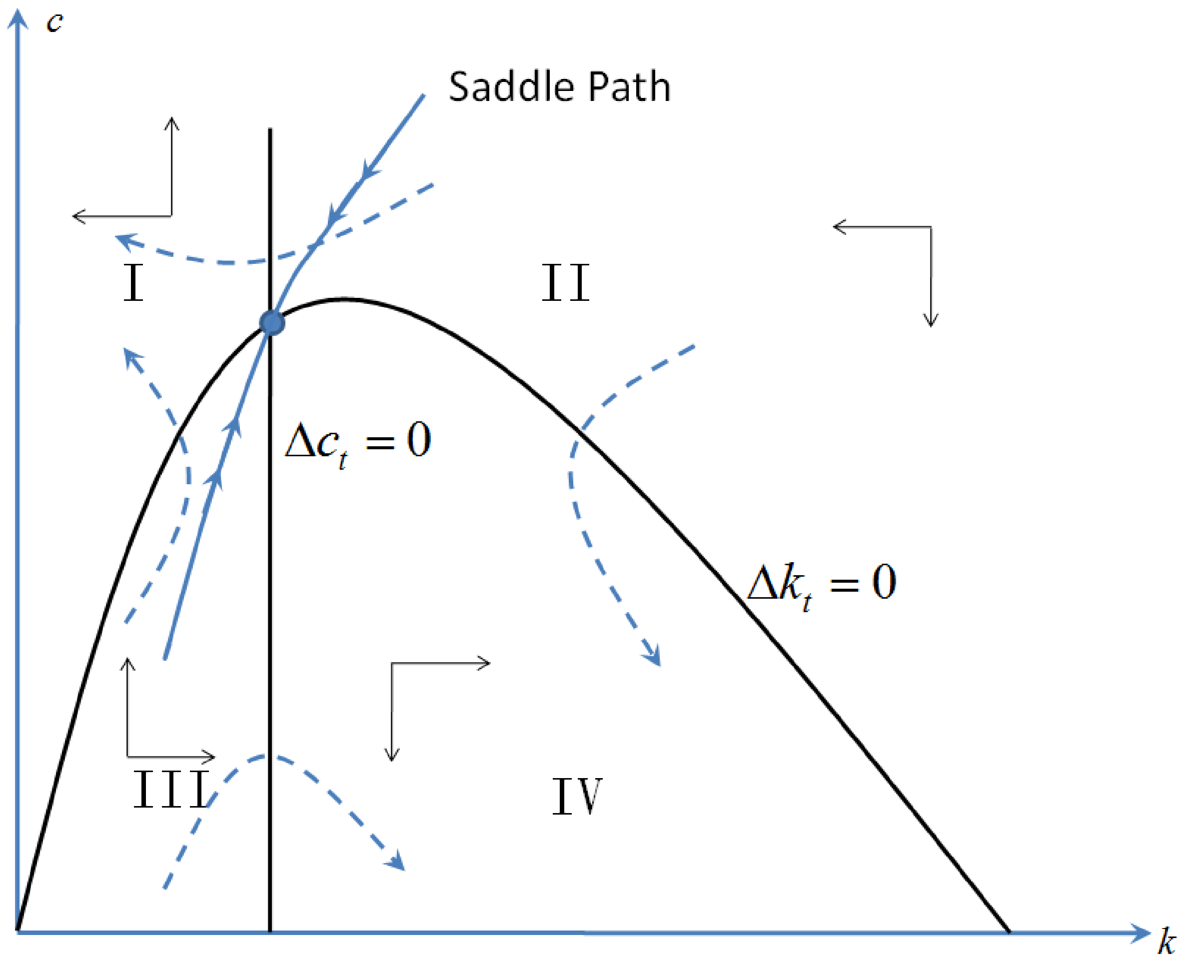
Area III: $c_t \uparrow$ and $k_t \uparrow$. The dynamics in this area is just opposite to those in Area II.

Area IV: $c_t \downarrow$ and $k_t \uparrow$. The dynamics in this area is just opposite to those in Area I.

According to the above dynamic analysis, there exists a unique path (solid line with arrows in Figure 1) such that from any points in this path the system will eventually converge to the steady state. We call this unique equilibrium path as the "saddle path".[3]

[3] Technically speaking, the difference equation system (18) and (19) just describes the dynamics of Δc_t and Δk_t instead of the levels of c_t and k_t. These two equations only provide necessary conditions for the optimal path of $\{c_t, k_t\}$. To find out the optimal path, we need to check $\lim_{t\to\infty} c_t$ and $\lim_{t\to\infty} k_t$ according to the phase diagram. The "saddle path" is the only path that ensures the sequences of c_t and k_t not diverge.

Figure 1: Phase diagram for c_t and k_t



1.6 Modified Golden Rule and Balanced Growth Path

Remember that the golden rule is defined as the steady-state (S-S) consumption at the maximum level. From (22), the capital stock at implied by the golden rule, k^{GR}, satisfies

$$f'\left(k^{GR}\right) = (1+n)(1+g) - (1-\delta). \tag{23}$$

However, the optimal steady-state capital k^* satisfies

$$f^{'}(k^*) = \frac{(1+n)(1+g)}{\beta} - (1-\delta). \tag{24}$$

Therefore, only if the discounting rate $\beta = 1$, the optimal S-S capital k^* is identical to the golden rule capital k^{GR}. It is easy to see that if $\beta < 1$, $k^{GR} > k^*$.[4] Moreover, the S-S saving rate in the Ramsey model is

$$\begin{aligned} s^* &= 1 - \frac{c^*}{y^*} \\ &= 1 - \frac{f(k^*) - [(1+n)(1+g) - (1-\delta)]\, k^*}{f(k^*)} \\ &= [(1+n)(1+g) - (1-\delta)] \frac{k^*}{f(k^*)} \\ &= \frac{(1+n)(1+g) - (1-\delta)}{(1+n)(1+g)/\beta - (1-\delta)} \alpha < \alpha. \end{aligned} \tag{25}$$

Last equality is due to the fact that $\frac{f'(k^*)k^*}{f(k^*)} = \alpha$ and $f^{'}(k^*) = \frac{(1+n)(1+g)}{\beta} - (1-\delta)$. The optimal S-S saving rate is less than the Golden-Rule saving rate if $\beta < 1$. The intuition is that keeping the maximum consumption c^{GR} at each period is not optimal because the household cares more about current period than the future ($\beta < 1$). Therefore, starting at $k = k^{GR}$ (higher than k^*), the household always has incentive to consume more than c^{GR} in the current period.

2 Transitional Dynamics: Further Discussions

2.1 A Special Case

We now provide a more rigorous discussion on the model dynamics. To simplify the math, we consider a special case where the capital is fully depreciated, i.e., $\delta = 1$. The resource constraint (13) can be simplified as

$$k_{t+1} = \frac{1}{(1+n)(1+g)} [f(k_t) - c_t]. \tag{26}$$

[4]This is because $f'(k) = \alpha k^{\alpha-1}$ decreases with k.

And (18) can be rewritten as

$$\frac{\Delta c_{t+1}}{c_t} = \frac{\beta}{(1+n)(1+g)} f'(k_{t+1}) - 1. \tag{27}$$

We employ guess-and-verify strategy to solve the model. In particular, we guess the optimal capital decision is linear in income, i.e., $k_{t+1} = sf(k_t)$, where s is an unknown coefficient to be determined. Then, from (26) we solve the consumption as

$$c_t = [1 - (1+n)(1+g)s] f(k_t). \tag{28}$$

Now we need to pin down s. To do so, we replace c_t in (27) with last equation and obtain

$$\frac{f(k_{t+1})}{f(k_t)} - 1 = \frac{\beta}{(1+n)(1+g)} f'(k_{t+1}) - 1. \tag{29}$$

Remember that $f(k) = k^\alpha$ and $f'(k) = \alpha k^{\alpha-1}$. We can solve k_{t+1} as

$$k_{t+1} = \frac{\alpha\beta}{(1+n)(1+g)} f(k_t). \tag{30}$$

Last equation confirms the linear form of our initial guess of k_{t+1} and we immediately have $s = \frac{\alpha\beta}{(1+n)(1+g)}$. The optimal consumption c_t follows

$$c_t = (1 - \alpha\beta) f(k_t). \tag{31}$$

From the above analysis, we can see that the Ramsey model degenerates to the Solow model when the depreciation rate is 1. So the transition dynamics in the Ramsey model replicate those in the Solow model. Moreover, the consumption rate is $1-\alpha\beta$, while in the Solow model under the Golden rule, the consumption rate is $1-\alpha$. As we discussed before, the discount factor $\beta < 1$ gives the household more incentive to consume in the current period, resulting in a higher consumption rate comparing that in under the Golden rule.

2.2 Demand Shock

2.2.1 Unexpected Changes of β

So far, we have discussed how to obtain the saddle path, along which the economy will eventually converge to the steady state (c^*, k^*). Given the fixed fundamental (no shocks, no changes of the values of parameters), the saddle path is a unique path that solves the optimization problem. Any initial (c_0, k_0) off the saddle path will eventually diverge, and the corresponding path is not optimal to the economy.

In this section, we will discuss the scenario that if there is an unexpected change of the fundamental, what will the economy respond. To take a concrete example, we discuss the change of the discounting rate β, which captures the demand shock. For a lower β, the consumer weights current consumption more than that in the future. Therefore, a decrease in β corresponds to a positive demand shock in current period.

Let's consider a time-varying β. We label it as β_t. We still use the special case of $\delta = 1$ to analyze the dynamics. Suppose that in the period 0, the economy is at the old steady state: (c^*, k^*). In the same period, there is a sudden permanent decrease in β_t, i.e, $\beta_t = \beta$ if $t = 0$ and $\beta_t = \beta^{new}$ for $t = 1, 2, 3...$ You may take the change of β_t as a surprise or an exogenous permanent shock. We denote the new steady state as (c^{**}, k^{**}). Under the new β^{new} (starting from period $t = 1$) the dynamic system will be changed to

$$\frac{\Delta c_{t+1}}{c_t} = \frac{\beta^{new}}{(1+n)(1+g)} f'(k_t) - 1, \tag{32}$$

$$\Delta k_{t+1} = \frac{1}{(1+n)(1+g)} \left[f(k_t) - (1+n)(1+g) k_t - c_t \right]. \tag{33}$$

The optimal path satisfies

$$k_{t+1} = \frac{\alpha \beta^{new}}{(1+n)(1+g)} f(k_t). \tag{34}$$

$$c_t = (1 - \alpha \beta^{new}) f(k_t). \tag{35}$$

To see the phase diagram, notice that the change of β_t will shift the curve of $\Delta c_t = 0$ to the left but keeps the curve $\Delta k_t = 0$ unchanged. Figure 2 illustrates the new phase diagram under β^{new}.

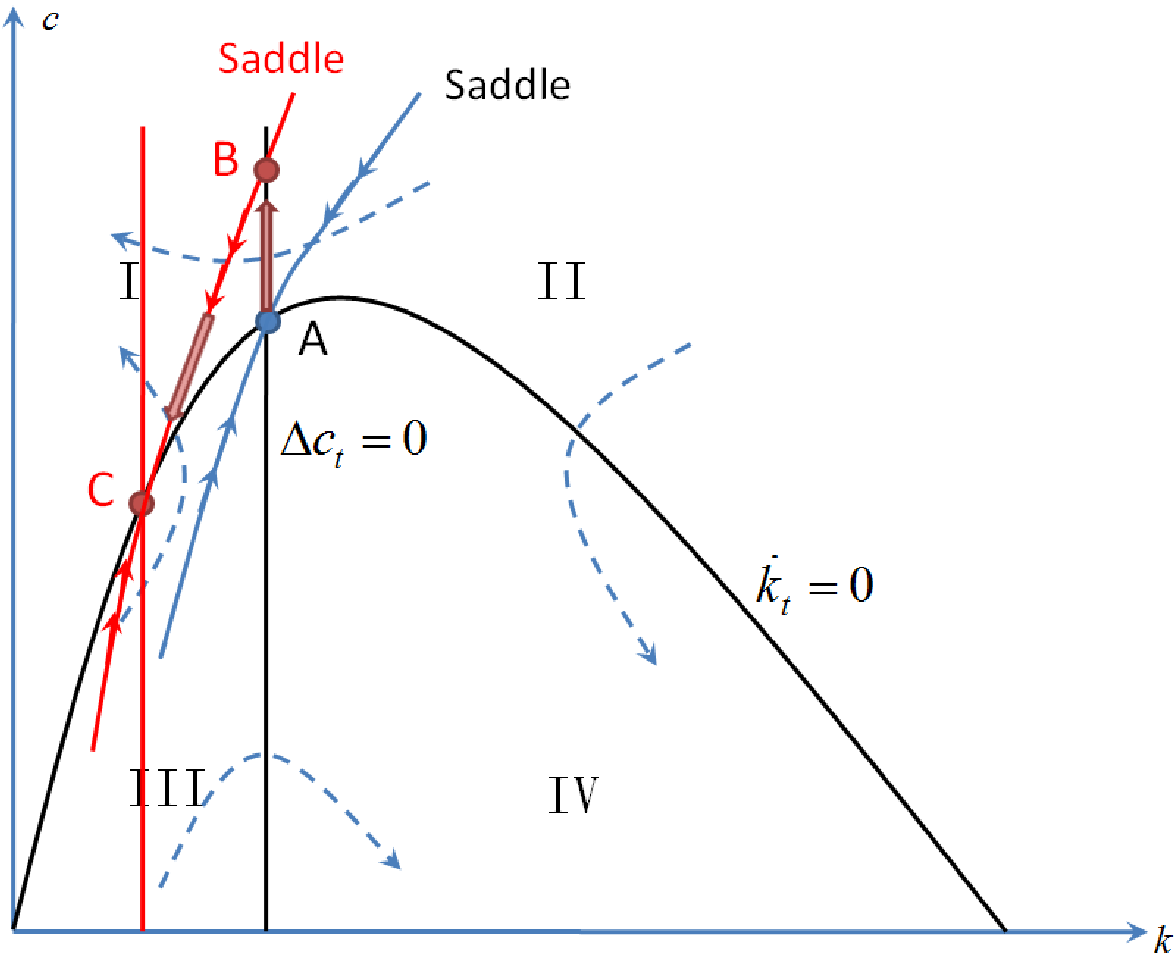
Note that the k_0 $(= k^*)$ does not change in the period 0, because k is a predetermined (or state) variable. Figure 4 illustrates the transition dynamics: $A \to B \to C$. When the fundamental changes, the consumption adjusts such that (c, k) will jump towards the new saddle path.[5] Thus, we call the variable c_t as control variable or jump variable.

2.2.2 Expected Changes of β

Suppose that in the period 0, the economy is at the old steady state (c^*, k^*). In the same period, there is an announcement saying that there will be a permanent decrease in β starting from $t = T$. That is, $\beta_t = \beta^{old}$ when $t < T$, and $\beta_t = \beta^{new}$, when $t \geq T$. We denote the new steady state as (c^{**}, k^{**}). To see how the economy responds to this news, we now solve the path through guess-and-verify strategy. Since now β_t is time varying, we guess $k_{t+1} = s_t f(k_t)$. Using the same

[5]In this case, the consumption immediately jumps to the saddle path, this is mainly due to the change of β is unexpected and permanent.

Figure 2: Transition dynamics under unexpected change of β_t



trick in the previous discussion yields

$$c_t = [1 - (1+n)(1+g)s_t] f(k_t), \tag{36}$$

$$k_{t+1} = \frac{\alpha\beta_t}{(1+n)(1+g)} \frac{1-(1+n)(1+g)s_t}{1-(1+n)(1+g)s_{t+1}} f(k_t). \tag{37}$$

Finally, to pin down s_t, we have

$$s_t = \frac{\alpha\beta_t}{(1+n)(1+g)} \frac{1-(1+n)(1+g)s_t}{1-(1+n)(1+g)s_{t+1}}. \tag{38}$$

Last equation provides a dynamic structure for the saving rate s_t. It takes a recursive relationship between s_t and s_{t+1}, i.e.,

$$s_t = g(s_{t+1}, \beta_t). \tag{39}$$

From this recursive structure, we can solve s_t by iterating last equation forwardly, i.e.,

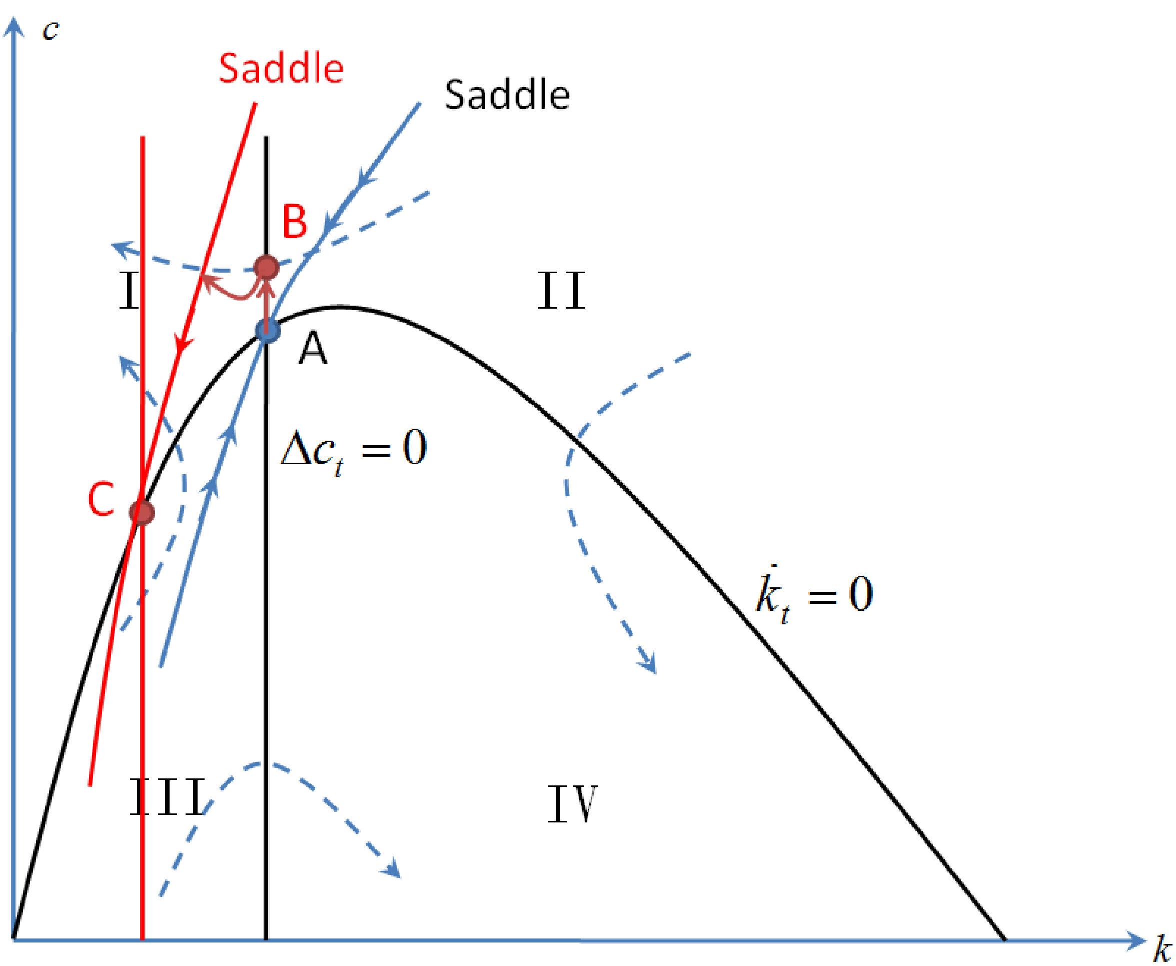
$$s_t = g\left(g\left(s_{t+2}, \beta_{t+1}\right), \beta_t\right) = g\left(g\left(g\left(s_{t+3}, \beta_{t+2}\right), \beta_{t+1}\right), \beta_t\right) = ... \tag{40}$$

You can image that eventually we will obtain a formula like[6]

$$s_t = h\left(\beta_t, \beta_{t+1}, \beta_{t+2}, ..., \lim_{t\to\infty} \beta_t; \lim_{t\to\infty} s_t\right). \tag{41}$$

The above equation provides a very important feature of the Ramsey model: when households making their optimal saving decisions, they are forward looking. Therefore, any change about the economic fundamental in the future (even though it does not happen right now) will affect the economic behaviors in the current period. News shock matters for the economy!

Figure 3: Transition dynamics under expected change of β_t



[6]To be more clear, we can linearize (38) around the steady state $(1+\rho)\hat{s}_t = \hat{\beta}_t + \rho\hat{s}_{t+1}$, where $\hat{s}_t = \frac{s_t - s}{s}$, $\hat{\beta}_t = \frac{\beta_t - \beta}{\beta}$, $\rho = \frac{(1+n)(1+g)}{1-\alpha\beta}$. Iterating the equation forwardly, we eventually have $\hat{s}_t = \sum_j \left(\frac{\rho}{1+\rho}\right)^j \hat{\beta}_{t+j}$.

Figure 3 illustrates the transition dynamics under an expected decrease in β_t. It shows that even though the change of β_t will occur in the future, the optimal transition for the economy is to jump in the initial period (remember that the β is still unchanged for all $t < T$), and keep moving under the old system until the news is realized, at that time (c_t, k_t) just arrives the new saddle path. The transition path in the Figure 3 is: $A \to B \to C$.

2.3 Government Tax

In the Ramsey model, we can easily introduce a government tax rate. We take the above special case as an example. We assume the government imposes an income tax τ on the household. So the disposable income for the household is $(1-\tau) f(k_t)$. Then the dynamic system becomes

$$k_{t+1} = \frac{1}{(1+n)(1+g)} \left[(1-\tau) f(k_t) - c_t\right], \tag{42}$$

$$\frac{\Delta c_{t+1}}{c_t} = \frac{\beta}{(1+n)(1+g)} (1-\tau) f'(k_{t+1}) - 1. \tag{43}$$

Modify the solution in the special case a bit, we obtain

$$k_{t+1} = \frac{\alpha\beta}{(1+n)(1+g)} (1-\tau) f(k_t), \tag{44}$$

$$c_t = (1-\alpha\beta)(1-\tau) f(k_t). \tag{45}$$

Last equations indicate that an increase in the income tax τ will reduce the capital and consumption.

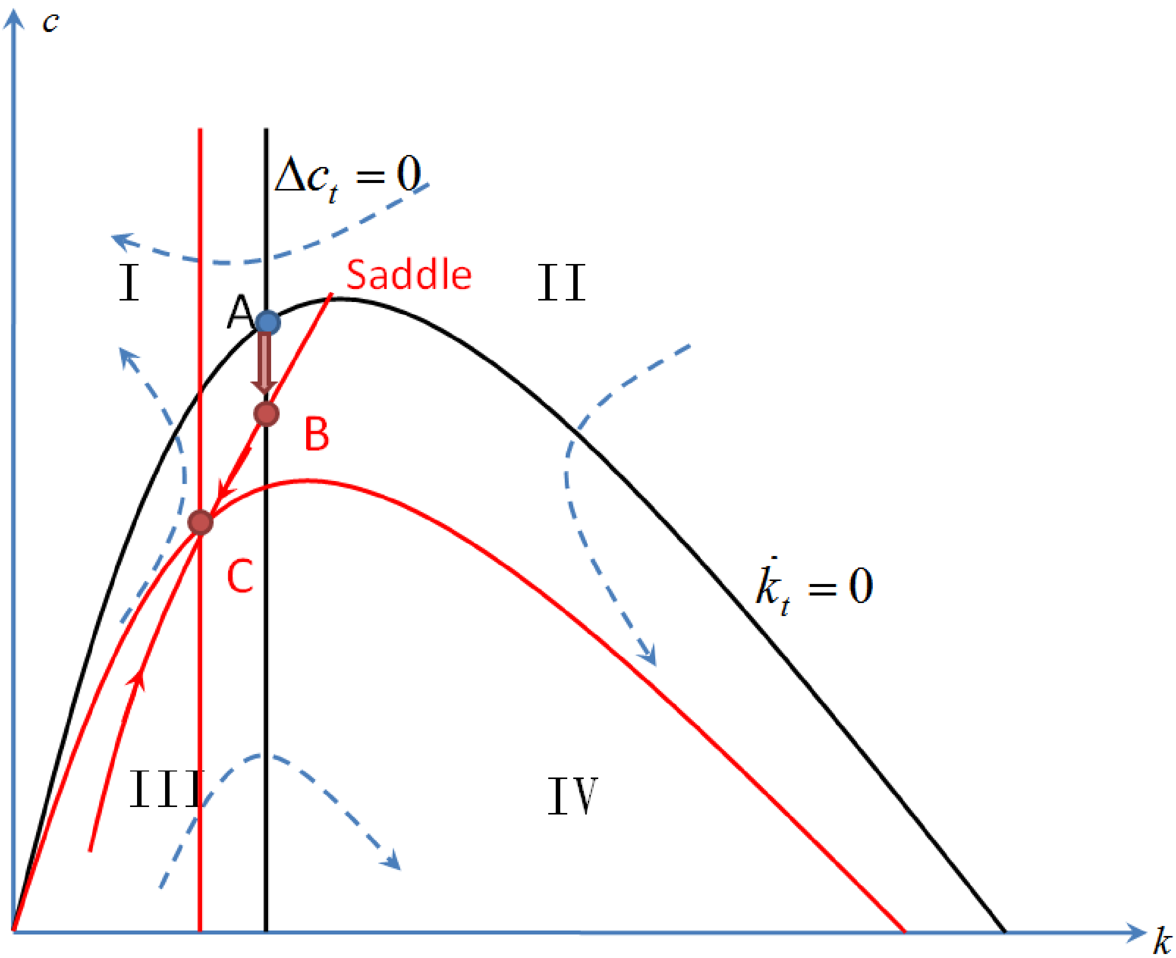
2.3.1 Expected Changes of τ

Now we use the phase diagram to discuss the dynamic impact of a change in tax rate on the aggregate economy. Notice that an increase in τ will shift the curves of $\Delta c_t = 0$ and $\Delta k_t = 0$ to the left. Figure 4 illustrates the new phase diagram.

Note that the k_0 $(= k^*)$ does not change in the period 0, because k is a predetermined (or state) variable. Figure 4 illustrates the transition dynamics: $A \to B \to C$. When the fundamental changes, the consumption adjusts such that (c, k) will jump towards the new saddle path.[7] Thus, we call the variable c_t as control variable or jump variable.

[7] In this case, the consumption immediately jumps to the saddle path, this is mainly due to the change of β is unexpected and permanent.

Figure 4: Transition dynamics under unexpected change of β_t



2.3.2 Expected Changes of τ

Suppose that in the period 0, the economy is at the old steady state (c^*, k^*). In the same period, there is an announcement saying that there will be a permanent decrease in τ starting from $t = T$. That is, $\tau_t = \tau^{old}$ when $t < T$, and $\tau_t = \tau^{new}$, when $t \geq T$. We denote the new steady state as (c^{**}, k^{**}). To see how the economy responds to this news, we now solve the path through guess-and-verify strategy. Since now τ_t is time varying, we guess $k_{t+1} = s_t (1 - \tau_t) f(k_t)$. Using the same trick in the previous discussion yields

$$c_t = [1 - (1+n)(1+g) s_t (1-\tau_t)] f(k_t), \tag{46}$$

$$k_{t+1} = \frac{\alpha\beta}{(1+n)(1+g)} \frac{(1-\tau_t)[1-(1+n)(1+g) s_t (1-\tau_t)]}{1-(1+n)(1+g) s_{t+1} (1-\tau_{t+1})} f(k_t). \tag{47}$$

Finally, to pin down s_t, we have

$$s_t = \frac{\alpha\beta}{(1+n)(1+g)} \frac{1-(1+n)(1+g)s_t(1-\tau_t)}{1-(1+n)(1+g)s_{t+1}(1-\tau_{t+1})}. \tag{48}$$

Last equation provides a dynamic structure for the saving rate s_t. It takes a recursive relationship between s_t and s_{t+1}, i.e.,

$$s_t = g(s_{t+1}, \tau_t, \tau_{t+1}). \tag{49}$$

From this recursive structure, we can solve s_t by iterating last equation forwardly, i.e.,

$$s_t = g(g(s_{t+2}, \tau_{t+1}, \tau_{t+2}), \tau_t, \tau_{t+1}) = g(g(g(s_{t+3}, \tau_{t+2}, \tau_{t+3}), \tau_{t+1}, \tau_{t+2}), \tau_t, \tau_{t+1}) = ... \tag{50}$$

You can image that eventually we will obtain a formula like

$$s_t = h\left(\tau_t, \tau_{t+1}, \tau_{t+2}, ..., \lim_{t\to\infty} \tau_t; \lim_{t\to\infty} s_t\right). \tag{51}$$

The above equation provides a very important feature of the Ramsey model: when households making their optimal saving decisions, they are forward looking. Therefore, any change about the economic fundamental in the future (even though it does not happen right now) will affect the economic behaviors in the current period. News shock matters for the economy!

Figure 5 illustrates the transition dynamics under an expected increase in τ_t. It shows that even though the change of τ_t will occur in the future, the optimal transition for the economy is to jump in the initial period (remember that the τ is still unchanged for all $t < T$), and keep moving under the old system until the news is realized, at that time (c_t, k_t) just arrives the new saddle path. The transition path in the Figure 5 is: $A \to B \to C$.

Figure 5: Transition dynamics under expected change of β_t