Lecture Notes 3: The Overlapping Generations Model

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The Ramsey-Cass-Koopmans model considers a representative household that lives infinite horizons. In many circumstances, however, the assumption of a representative household is not appropriate. One important set of circumstances that may require departure from this assumption is in the analysis of an economy in which new households are born over time. The arrival of new households in the economy is not only a realistic feature, but it also introduces a range of new economic interactions. In particular, decisions made by older generations will affect the prices faced by younger generations. These economic interactions have no counterpart in the neoclassical growth model. They are most succinctly captured in the overlapping generations (OLG) models introduced and studied by Paul Samuelson and later by Peter Diamond. The OLG model considers infinite agents who only live finite periods. In particular, new individuals are continually being born, and old individuals are continually dying.

The OLG model is useful for a number of reasons. First, it captures the potential interaction of different generations of individuals in the marketplace. Second, it provides a tractable alternative to the infinite-horizon representative agent models. Third, some of the key implications are different from those of the neoclassical growth model (e.g. dynamic inefficiency). Finally, the OLG model provides a flexible framework to study the effects of macroeconomic policies such as national debt and social security.

1 Economic Environment

In this economy, time is discrete and runs to infinity. Each individual lives two periods. For the generation born in period t, they live for period t and $t+1$. In period t, they are young generation, and become old generation in period $t+1$. As individuals live only two periods, the economy always have two generations in any period. L_t individuals are born in period t. As in Ramsey model, population grows at rate n, i.e.,

$$L_t = (1+n)\,L_{t-1}. \tag{1}$$

Thus, there are L_t young generation and L_{t-1} $(= L_t/\,(1+n))$ old generation.

1.1 Consumers

Each consumer supplies 1 unit of labor at wage rate W_t when he/she is young and divides the labor income between first-period consumption and saving with interest rate R_t. In the second period, the individual simply consumes the saving and any interest he/she earns. Let c_{1t} and c_{2t} denote the consumption in period t of young and old individuals. A representative individual born in period t solves

$$\max_{\{c_{1t},c_{2t+1}\}} \frac{c_{1t}^{1-\theta}}{1-\theta} + \beta \frac{c_{2t+1}^{1-\theta}}{1-\theta} \tag{2}$$

subject to budget constraint

$$\begin{aligned} c_{1t} + s_t &\leq W_t, & (3)\\ c_{2t+1} &\leq R_{t+1} s_t. & (4) \end{aligned}$$

The above problem can be written more compactly as

$$\max_{s_t} \frac{(W_t - s_t)^{1-\theta}}{1-\theta} + \beta \frac{(R_{t+1} s_t)^{1-\theta}}{1-\theta}, \tag{5}$$

and consumptions are give by

$$\begin{aligned} c_{1t} &= W_t - s_t, & (6)\\ c_{2t+1} &= R_{t+1} s_t. & (7) \end{aligned}$$

First order condition for the optimal saving is

$$(W_t - s_t)^{-\theta} = \beta R_{t+1} (R_{t+1} s_t)^{-\theta}. \tag{8}$$

Thus the optimal saving s_t is given by

$$s_t = s(R_{t+1}) W_t. \tag{9}$$

where $s(R_t) = \frac{1}{1+\beta^{-\frac{1}{\theta}} R_{t+1}^{1-\frac{1}{\theta}}}$ indicates the saving rate. Note that, for $\theta = 1$ (the utility is logarithm), the saving rate is just a constant $\frac{\beta}{1+\beta}$. Later we will show that in this case the OLG model is equivalent to the Solow model with saving rate $\beta/(1+\beta)$. Moreover, optimal consumptions are given by

$$c_{1t} = [1 - s(R_{t+1})] W_t, \tag{10}$$

$$c_{2t+1} = R_{t+1} s(R_{t+1}) W_t. \tag{11}$$

1.2 Firms

A representative firm hires labor L_t and rents capital K_t to produce final goods according to the production function $Y_t = F(K_t, A_t L_t)$, where the technology A_t is assumed to follow

$$A_t = (1+g)^t A_{t-1}. \tag{12}$$

We assume that the capital is fully depreciated. The firm aims to maximize the profit by choosing L_t and K_t. The optimization problem is

$$\max_{\{L_t, K_t\}} F(K_t, A_t L_t) - W_t L_t - R_t K_t. \tag{13}$$

The first order conditions w.r.t. $\{L_t, K_t\}$ are given by

$$R_t = F_K(K_t, A_t L_t), \tag{14}$$

$$W_t = F_{AL}(K_t, A_t L_t) A_t. \tag{15}$$

We assume the production function is constant return to scale. Let $f(k) = F\left(\frac{K}{AL}, 1\right)$, where $k = \frac{K}{AL}$. The input demands can be expressed as

$$R_t = f'(k_t), \tag{16}$$

$$W_t = \left[f(k_t) - f'(k_t) k_t\right] A_t. \tag{17}$$

1.3 Competitive Equilibrium

In the competitive equilibrium, consumers and firms achieve the individual optimum. Each market clears. In particular, capital market clearing condition implies

$$K_{t+1} = L_t s(R_{t+1}) W_t. \tag{18}$$

According to (16) and (17), the last equation can be rewritten as

$$k_{t+1} = \frac{1}{(1+g)(1+n)} s\left(f'(k_{t+1})\right) \left[\frac{f(k_t) - f'(k_t) k_t}{f(k_t)}\right] f(k_t). \tag{19}$$

The above equation fully describes the dynamics of capital stock.

2 Dynamics

- **Special Case:** $\theta = 1$.

  Assume that $\theta = 1$ (utility is logarithm) and the production function takes Cobb-Douglas form, i.e., $F(K, AL) = K^\alpha (AL)^{1-\alpha}$. The saving rate in this case is $s(f'(k_{t+1})) = \frac{\beta}{1+\beta}$. Equation (19) can be reduced into

$$k_{t+1} = \frac{1}{(1+g)(1+n)} \frac{\beta}{1+\beta} (1-\alpha) k_t^\alpha. \tag{20}$$

  Note that (20) essentially has the same form as the one derived from the Solow model. Hence, when utility function takes logarithm form, the OLG model is degenerated to the Solow model.

- **General Case**

  Once we relax the assumptions of logarithmic utility and Cobb-Douglas production technology, a wide range of behaviors of the economy are possible, see Romer's textbook page 84-87.

3 Dynamic Inefficiency

Even though in the OLG model, competitive equilibrium (CE) is achieved, it turns out that the CE allocation is not Pareto efficient. To see this, let us discuss the capital stock at the steady state. For simplicity, we still consider the special case where $\theta = 1$ and $f(k) = k^{\alpha}$.

From (20), we can obtain the steady-state capital stock k^* for the competitive equilibrium from

$$f'(k^*) = \alpha (k^*)^{\alpha-1} = (1+g)(1+n)\left(\frac{1+\beta}{\beta}\frac{\alpha}{1-\alpha}\right). \tag{21}$$

Now consider a social planner's problem:

$$\begin{aligned} & \max_{\{c_{1t},c_{2t}\}} \sum_{t=0} \beta^t \left(L_t \frac{c_{1t}^{1-\theta}}{1-\theta} + L_{t-1}\phi \frac{c_{2t}^{1-\theta}}{1-\theta} \right) \\ = & \max_{\{c_{1t},c_{2t}\}} \sum_{t=0} \left[\beta (1+n)(1+g)\right]^t \left(\frac{\tilde{c}_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+n}\phi \frac{\tilde{c}_{2t}^{1-\theta}}{1-\theta} \right) \end{aligned} \tag{22}$$

where $\phi > 0$ is the weight that social planner puts on the old generation, $\tilde{c}_{1t} = c_{1t}/A_t$, $\tilde{c}_{2t} = c_{2t}/A_t$. The resource constraint is

$$L_t c_{1t} + L_{t-1} c_{2t} + K_{t+1} = Y_t. \tag{23}$$

Detrending both side with $A_t L_t$ gives us

$$\tilde{c}_{1t} + \frac{\tilde{c}_{2t}}{1+n} + (1+n)(1+g) k_{t+1} = f(k_t). \tag{24}$$

FOCs w.r.t $\{\tilde{c}_{1t}, \tilde{c}_{2r}, k_{t+1}\}$ are given by

$$\tilde{c}_{1t}^{-\theta} = \phi \tilde{c}_{2t}^{-\theta} = \lambda_t, \tag{25}$$

$$(1+n)(1+g)\lambda_t = \beta \lambda_{t+1} f'(k_{t+1}). \tag{26}$$

In the steady state, we have

$$f'\left(\bar{k}\right) = \frac{(1+n)(1+g)}{\beta}. \tag{27}$$

Comparing (21) with (27), the capital stock in competitive equilibrium is Pareto optimal only if

$$\frac{(1+\beta)\alpha}{1-\alpha} = 1. \tag{28}$$

Therefore, in general, the competitive equilibrium in the OLG model is not Pareto optimal.

3.1 An Illustrative Example

The dynamic inefficiency in the OLG model indicates that the first welfare theorem does not apply here. Why? The reason is that the theorem requires finite number of households, whereas there are infinite number of households here. To give an intuitive illustration, let us consider following simple example (Acemoglu, 2009, page 328-329).

Consider the following static economy with a countably infinite number of households, each denoted by $i \in N$, and a countably infinite number of commodities, denoted by $j \in N$. Assume that all households behave competitively. Household i has preferences given by

$$u_i = c_i^i + c_{i+1}^i$$

where $c_j^i \geq 0$ denotes the consumption of the jth type of commodity by household i. These preferences imply that household i enjoys the consumption of the commodity with the same index as its own and the next indexed commodity (e.g., the household indexed by 3 only derives utility from the consumption of goods indexed by 3 and 4). The endowment vector ω of the economy is as follows: each household has one unit endowment of the commodity with the same index as its own (i.e., for i the endowment is $\omega_i = 1$ unit of goods i). Let us choose the price of the first commodity as the numeraire, so that $p_0 = 1$. The competitive equilibrium is defined as the combination of price vector $\{p_j\}$ and allocation $\{c_j^i\}$ such that under these prices, each individual maximizes own utility. The individual i's optimization problem is

$$\max_{\{c_i^i, c_{i+1}^i\}} u_i = c_i^i + c_{i+1}^i$$

subject to

$$p_i c_i^i + p_{i+1} c_{i+1}^i \leq p_i \times 1.$$

It is easy to show that $p_j = 1$ for all j and no trade among individuals (i.e., $c_i^i = 1$, $c_j^i = 0$) is competitive equilibrium. However, the CE allocation is not Pareto optimal. To see this, let us consider following allocation. Each household $i < i'$ consumes one unit of good i. Household i' consumes one unit of good i' and one unit of good $i'+1$. Finally, household $i > i'$ consumes one unit of good $i+1$. In other words, household i' consumes its own endowment and that of household $i'+1$. While all other households, indexed $i > i'$, consume the endowment of the neighboring household, $i+1$ (while the consumption bundles of all households $i < i'$ are the same as in competitive equilibrium). In this allocation, all households with $i \neq i'$ are as well off as in the competitive equilibrium, and household i' is strictly better off.

4 The OLG Model with Credit Market Imperfection

One application of the OLG model is to introduce credit market imperfection. In this section, we will discuss Masuyama's (2009) model. The consumer's behavior is the same as the standard OLG model, the only difference is that the firm is subject to credit constraint. Let us first specify the consumer's problem. For simplicity, we assume that the population is fixed, $L_t = L_{t-1} = 1$, and there is no technology progress, $A_t = 1$.

A representative consumer born in period t solves

$$\max_{\{c_{1t}, c_{2t+1}\}} V\left(c_{1t}\right) + c_{2t+1} \tag{29}$$

subject to budget constraint

$$\begin{aligned} c_{1t} + s_t &\leq W_t, & (30) \\ c_{2t+1} &\leq r_t s_t, & (31) \end{aligned}$$

where W_t is the wage rate and r_t is the interest rate. It can be shown that the optimal saving is given by

$$s_t = W_t - \left(V'\right)^{-1}\left(r_t\right). \tag{32}$$

To capture the behaviors in credit market, we assume that there are representative entrepreneurs with unit mass. The entrepreneurs live two periods and with endowment ω. In the first period, they can run non-divisible investment projects, which convert one unit investment to R unit of capital in the second period by borrowing $1-\omega$ at the interest rate r_t. In the second period, the capital is used to produce consumption goods with production function $y_{t+1} = f\left(k_{t+1}\right)$. Thus, the revenue of one project started in period t is $Rf'\left(k_{t+1}\right)$. If the entrepreneur does not invest, they can lend their money to other investors to earn the interest.

Let us first consider the problem facing by the entrepreneur who make investment. As the credit market is imperfect, to finance their investments, the entrepreneurs are subject to following credit constraint

$$r_t\left(1-\omega\right) \leq \lambda R f'\left(k_{t+1}\right). \tag{33}$$

This constraint means that no more than a fraction, λ, of the project revenue can be pledged to the lenders for the interest payment. The parameter λ thus captures the financial development. For simplicity, we assume that $\frac{\lambda}{1-\omega} < 1$, and the constraint (33) holds with equality. That is,

$$Rf'\left(k_{t+1}\right) = \frac{1-\omega}{\lambda} r_t. \tag{34}$$

The above equation describes the capital demand. Compared to the perfect economy, the financial friction introduces a wedge $\frac{1-\omega}{\lambda}$ (> 1) between the interest rate and the marginal product of capital.

The wedge captures the extent of distortion in the credit market. Moreover, as one project will generate R units of capital, the capital supply is given by

$$\begin{aligned} k_{t+1} &= Rs_t, \\ &= R\left[W_t - \left(V'\right)^{-1}(r_t) + \omega\right]. \end{aligned} \tag{35}$$

The second line is due to the saving function (32). Note that from the labor market clearing condition, the wage rate is given by

$$W_t = f(k_t) - k_t f'(k_t). \tag{36}$$

Finally, the full dynamic system is given by (34), (35) and (36).

Assume production function $f(k) = k^\alpha$, and utility function $V(c) = \log(c)$. The system can be reduced to

$$k_{t+1} + \frac{1-\omega}{\alpha\lambda} k_{t+1}^{1-\alpha} = R\left[(1-\alpha)k_t^\alpha + \omega\right]. \tag{37}$$

If financial market is more advanced, i.e., λ increases, it can be shown that the steady state capital k^* increases.

We now use a numerical example to study the quantitative effect of financial development on the economic growth. First, we calibrate the parameters as follows

$$\alpha = 0.5,\ \omega = 0.75,\ R = 1.5.$$

We assume that in the period 1, the economy stays at the steady state with $\lambda = 0.5$, where steady state capital is k^*; In the period 2, the financial market develops, and λ increases to 0.6. Denote the new steady-state capital as k^{**}. It is easy to show that $k^{**} > k^*$. According to (37), we can compute the transitional path of capital k_t. Figure 1 plots the paths of capital and output.

Figure 1. Effects of Financial Development on the Growth