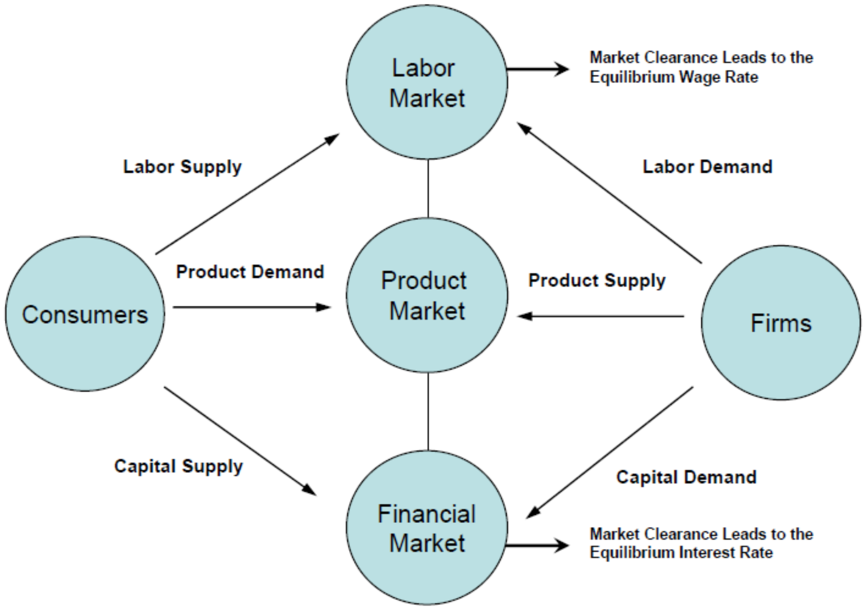
Macroeconomic Theory
——From the Classical to the New Keynesian

DENG Yanfei
dengyf@fudan.edu.cn
https://weibo.com/dengyfman0616

April 22, 2020

Contents

1 The RBC Model——Its Extensions



A Topology of Macroeconomics [1]

1.1 A Note on Economic Growth

$$
\begin{aligned}
N_t &= (1+g_n)N_{t-1}, \quad N_t: \text{ The population} \\
&= (1+g_n)^t. \quad \text{with } N_0 \equiv 1. \\
Z_t &= (1+g_z)Z_{t-1}, \\
&= (1+g_z)^t. \quad \text{with } Z_0 \equiv 1.
\end{aligned}
$$

$$
\begin{aligned}
K_{t+1} &= I_t + (1-\delta)K_t. \\
Y_t &= A_t K_t^{\alpha}(Z_t N_t L_t)^{1-\alpha}, \quad L_t \in [0,1]: \text{ work-hours per person} \\
\frac{Y_t}{Z_t N_t} &= \frac{A_t K_t^{\alpha}(Z_t N_t L_t)^{1-\alpha}}{(Z_t N_t)^{\alpha}(Z_t N_t)^{1-\alpha}}, \\
&= A_t \tilde{K}_t^{\alpha} L_t^{1-\alpha}, \\
\hat{a}_t &= \rho_a \hat{a}_{t-1} + \epsilon_t^a \quad \Leftrightarrow \quad \ln A_t - \ln A = \rho_a(\ln A_{t-1} - \ln A) + \epsilon_t^a, \quad \epsilon_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma_a^2).
\end{aligned}
$$

The following are per capita variables:

$$
\begin{aligned}
\bar{C}_t &= \frac{C_t}{N_t} = \frac{C_t}{(1+g_n)^t}, \\
\bar{K}_t &= \frac{K_t}{N_t} = \frac{K_t}{(1+g_n)^t}, \\
\bar{I}_t &= \frac{I_t}{N_t} = \frac{I_t}{(1+g_n)^t}, \\
\bar{Y}_t &= \frac{Y_t}{N_t} = \frac{Y_t}{(1+g_n)^t}, \\
\bar{\Pi}_t &= \frac{\Pi_t}{N_t} = \frac{\Pi_t}{(1+g_n)^t}, \\
\bar{G}_t &= \frac{G}{N_t} = \frac{G}{(1+g_n)^t}, \\
\bar{T}_t &= \frac{T}{N_t} = \frac{T}{(1+g_n)^t},
\end{aligned}
$$

[1] Source: Jenny Xu, Lecture Notes, HKUST

and the following are per effective unit of labor variables

$$
\begin{aligned}
\tilde{C}_t &= \frac{C_t}{Z_t N_t} = \frac{C_t}{(1+g_z)^t(1+g_n)^t} = \frac{\bar{C}}{(1+g_z)^t}, \\
\tilde{K}_t &= \frac{K_t}{Z_t N_t} = \frac{K_t}{(1+g_z)^t(1+g_n)^t} = \frac{\bar{K}}{(1+g_z)^t}, \\
\tilde{I}_t &= \frac{C_t}{Z_t N_t} = \frac{I_t}{(1+g_z)^t(1+g_n)^t} = \frac{\bar{I}}{(1+g_z)^t}, \\
\tilde{Y}_t &= \frac{Y_t}{Z_t N_t} = \frac{Y_t}{(1+g_z)^t(1+g_n)^t} = \frac{\bar{Y}}{(1+g_z)^t}, \\
\tilde{\Pi}_t &= \frac{\Pi_t}{Z_t N_t} = \frac{\Pi_t}{(1+g_z)^t(1+g_n)^t} = \frac{\bar{\Pi}}{(1+g_z)^t}, \\
\tilde{G}_t &= \frac{G_t}{Z_t N_t} = \frac{G_t}{(1+g_z)^t(1+g_n)^t} = \frac{\bar{G}}{(1+g_z)^t}, \\
\tilde{T}_t &= \frac{T_t}{Z_t N_t} = \frac{T_t}{(1+g_z)^t(1+g_n)^t} = \frac{\bar{T}}{(1+g_z)^t}.
\end{aligned}
$$

1.2 The Representative Household

$$
\max_{\tilde{C}_t, L_t, \tilde{K}_{t+1}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{\tilde{C}_t^{1-\sigma} - 1}{1-\sigma} + \varphi \ln(1 - L_t) + \psi \tilde{G}_t \right],
$$

$$
\text{s.t.} \quad \tilde{C}_t + \tilde{I}_t = W_t L_t + R_t \tilde{K}_t - \tilde{T}_t, \quad \forall t,
$$

$$
(1+g_z)(1+g_n)\tilde{K}_{t+1} = \tilde{I}_t + (1-\delta)\tilde{K}_t \quad \Leftrightarrow \quad \frac{K_{t+1}}{Z_{t+1}N_{t+1}} \frac{Z_{t+1}}{Z_t} \frac{N_{t+1}}{N_t} = \frac{I_t}{Z_t N_t} + (1-\delta)\frac{K_t}{Z_t N_t} \quad \forall t, \quad \text{where } \tilde{K}_0 > 0 \text{ given},
$$

$$
\tilde{G}_t = \tilde{T}_t, \quad \forall t,
$$

$$
\lim_{t\to\infty} \mathbb{E}_0 \beta^t \lambda_t \tilde{K}_{t+1} = 0,
$$

$$
\tilde{G}_t = G e^{\hat{g}_t}, \qquad \text{with } \hat{g}_t = \rho_g \hat{g}_{t-1} + \epsilon_t^g, \quad \epsilon_t^g \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma_g^2),
$$

$$
\left.\begin{aligned} \tilde{C}_t &> 0 \\ \tilde{K}_t &> 0 \\ L_t &\in (0,1] \end{aligned}\right\} \text{exclude corner solutions and ignore non-negativity constraints.}
$$

$$
\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{\tilde{C}_t^{1-\sigma} - 1}{1-\sigma} + \varphi \ln(1-L_t) + \psi \tilde{G}_t + \lambda_t \left[W_t L_t + R_t \tilde{K}_t - \tilde{G}_t - \tilde{C}_t - (1+g_z)(1+g_n)\tilde{K}_{t+1} + (1-\delta)\tilde{K}_t \right] \right\}.
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \tilde{C}_t} &= 0 \quad \Rightarrow \\
\frac{\partial \mathcal{L}}{\partial L_t} &= 0 \quad \Rightarrow \\
\frac{\partial \mathcal{L}}{\partial \tilde{K}_{t+1}} &= 0 \quad \Rightarrow
\end{aligned}
$$

1.3 The Representative Firm

$$
\max_{L_t, \tilde{K}_t} \tilde{\Pi}_t = \tilde{Y}_t - W_t L_t - R_t \tilde{K}_t,
$$

$$
\text{s.t.} \quad \tilde{Y}_t = A_t F(\tilde{K}_t, L_t) = A_t \tilde{K}_t^{\alpha} L_t^{1-\alpha},
$$

$$
A_t = A e^{\hat{a}_t}, \qquad \text{where } \hat{a}_t = \rho_a \hat{a}_{t-1} + \epsilon_t^a, \quad \epsilon_t^a \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma_a^2).
$$

$$
\tilde{\Pi}_t = A_t \tilde{K}_t^{\alpha} L_t^{1-\alpha} - W_t L_t - R_t \tilde{K}_t.
$$

$$
\begin{aligned}
\frac{\partial \tilde{\Pi}_t}{\partial L_t} &= 0 \quad \Rightarrow \\
\frac{\partial \tilde{\Pi}_t}{\partial \tilde{K}_t} &= 0 \quad \Rightarrow
\end{aligned}
$$

1.4 Maket Clearing

① Labor market

$$L^{\text{supply}} = L^{\text{demand}} \quad \Rightarrow \tag{1}$$

② Capital market

$$\tilde{K}^{\text{supply}} = \tilde{K}^{\text{demand}} \quad \Rightarrow \tag{2}$$

③ Goods market

$$\tilde{C}_t + \tilde{I}_t + \tilde{G}_t = \tilde{Y}_t \quad \Rightarrow \quad \tilde{C}_t + (1+g_z)(1+g_n)\tilde{K}_{t+1} - (1-\delta)\tilde{K}_t + Ge^{\hat{g}_t} = Ae^{\hat{a}_t}\tilde{K}_t^{\alpha}L_t^{1-\alpha}. \tag{3}$$

1.5 Competitive Rational Expectations Equilibrium

Households and Firms are price takers and have rational expectations.

1) describe the solution implicitly

An equilibrium is an allocation $\{\tilde{C}_t, L_t, \tilde{K}_{t+1}\}_{t=0}^{\infty}$ where
given the exogenous state variables $\{\hat{g}_t, \hat{a}_t\}_{t=0}^{\infty}$ and $\tilde{K}_0$ (note that $\tilde{K}_t$ is a endogeous state variable),

the representative household maximize their utility,
the representative firm maximize their profits, and
the markets are clear.

$$\left.\begin{array}{l} (3) \Rightarrow \tilde{C}_t = \to (1) \Rightarrow L_t = \\ \qquad\qquad\qquad\qquad\quad \downarrow \\ (3) \Rightarrow \tilde{C}_t = \to (2) \to \end{array}\right\} \Rightarrow \quad \tilde{K}_{t+2} = \mathscr{F}\tilde{K}_{t+1} + \mathscr{E}\tilde{K}_t + \mathscr{D} \text{ (a 2nd-order difference equation) with LOMs+TV/No-Ponzi-C.}$$

2) solve the system explicitly

$$\begin{cases} \text{labor supply equation} \\ \text{labor demand equation} \\ \text{production function} \\ \text{goods clearing condition} \\ \text{consumption Euler equation} \\ \text{investment equation} \end{cases} \Leftrightarrow \begin{cases} L^{\text{supply}} = L^{\text{demand}}, \\ \tilde{K}^{\text{supply}} = \tilde{K}^{\text{demand}}, \\ \tilde{C}_t + (1+g_z)(1+g_n)\tilde{K}_{t+1} - (1-\delta)\tilde{K}_t + Ge^{\hat{g}_t} = Ae^{\hat{a}_t}\tilde{K}_t^{\alpha}L_t^{1-\alpha}, \\ \hat{g}_t = \rho_g \hat{g}_{t-1} + \epsilon_t^g, \\ \hat{a}_t = \rho_a \hat{a}_{t-1} + \epsilon_t^a. \end{cases} \Leftrightarrow \begin{cases} \text{choice variables: } \tilde{C}_t, L_t, \tilde{K}_{t+1}, \\ \text{endogenous state variables: } \tilde{K}_t, \\ \text{exogenous state variables: } \hat{g}_t, \hat{a}_t. \end{cases}$$

We have a system with the same number of equations as choice variables, however, there are two problems:
Ⓐ These equations are nonlinear;
Ⓑ The Euler equation is forward looking and the law of motion (LOM) of capital is backward looking.

There are three ways out:
ⓐ Use phase diagram → The solution is principle "global";
ⓑ Turn to numberical solution method → The solution is "global";
ⓒ Take linear approximations aroud the steady state → The solution is often "locally".

A first-order Taylor approximation (disadvantages + advantages)

Disadvantages:

Advantages: locally very accurate, provided well-defined approximation point and policy function is sufficiently smooth in the approximation point.

Point of approximation: deterministic steady state, i.e., resting point of the system without shocks for $t \to \infty$ and where agents take the absence of uncertainty into account:

$$\hat{x} = x_t - x = \ln X_t - \ln X = \frac{X_t - X}{X} \quad \Leftrightarrow \quad X_t = Xe^{\hat{x}_t} \approx X + Xe^{\hat{x}_t}\big|_{\hat{x}_t=0}(\hat{x}_t - 0) = X + X\hat{x}_t.$$

1.6 Linearization

I encourage the reader to do it by yourselves.

Having linearized the system, there is still one problem (dynamic+forward looking+backward looking) to overcome. To solve it, we can turn to:
① Method of undetermined coefficients;
② Diagonalization of the transition matrix.

1.7 The Rational-Expectations Equilibrium Determinacy

I) Solution Using Jump Variables (cf. McCandless, 2008, pp.104-)

$$\begin{cases} \text{The labor supply} = \text{the labor demand}(\hat{w}_t) & \Rightarrow \\ \text{The rental on capital}(\hat{r}_t) \quad \Rightarrow \\ \text{The production function} \quad \Rightarrow \\ \text{The budget constraints+Euler theorem}(\hat{y}_t)\text{+LOM of capital} & \Rightarrow \\ \text{The consumption Euler equation} \quad \Rightarrow \end{cases}$$

The **system** with 5 endogenous variables: $\hat{k}_{t+1}, \hat{y}_t, \hat{c}_t, \hat{l}_t, \hat{r}_t$

$$\begin{cases} \mathbf{x}_t \equiv [\hat{k}_{t+1}], & \leftarrow \text{the jump variable} \\ \mathbf{y}_t \equiv [\hat{y}_t, \hat{c}_t, \hat{l}_t, \hat{r}_t]', & \leftarrow \text{the endogenous state variables} \\ \mathbf{z}_t \equiv [\hat{g}_t, \hat{a}_t]'. & \leftarrow \text{the exogenous variables} \end{cases}$$

The linear version of the system can be written as

$$\begin{cases} \mathbf{0} = \mathcal{A}\mathbf{x}_t + \mathcal{B}\mathbf{x}_{t-1} + \mathcal{C}\mathbf{y}_t + \mathcal{D}\mathbf{z}_t, \\ \mathbf{0} = \mathbb{E}_t[\mathcal{F}\mathbf{x}_{t+1} + \mathcal{G}\mathbf{x}_t + \mathcal{H}\mathbf{x}_{t-1} + \mathcal{J}\mathbf{y}_{t+1} + \mathcal{K}\mathbf{y}_t + \mathcal{L}\mathbf{z}_{t+1} + \mathcal{M}\mathbf{z}_t], \\ \mathbf{z}_{t+1} = \mathcal{N}\mathbf{z}_t + \boldsymbol{\epsilon}_{t+1}, \quad \text{with } \mathbb{E}_t\boldsymbol{\epsilon}_{t+1} = \mathbf{0}. \end{cases}$$

The solution for this economy is a set of matrices

$$\begin{cases} \mathbf{x}_t = \mathcal{P}\mathbf{x}_{t-1} + \mathcal{Q}\mathbf{z}_t, \\ \mathbf{y}_t = \mathcal{R}\mathbf{x}_{t-1} + \mathcal{S}\mathbf{z}_t. \end{cases}$$

If the matrix $\mathcal{C}$ is of full rank (i.e., a well-defined inverse, $\mathcal{C}^{-1}$) and equilibrium laws of motion exist, they must fulfill (cf. Uhlig, 1999, A Toolkit for Analysing Nonlinear Dynamic Stochastic Models Easily):

$$\begin{cases} \mathbf{0} = ()\mathcal{P}^2 - ()\mathcal{P} - \cdots \\ \mathcal{R} = -\mathcal{C}^{-1}(\mathcal{A}\mathcal{P} + \mathcal{B}) \\ \mathcal{Q} = \\ \mathcal{S} = \end{cases}$$

II) Method of Undetermined Coefficients

$$L^{\text{supply}} = L^{\text{demand}} \quad \Rightarrow \quad L_t = \ \rightarrow \ \begin{cases} \tilde{C}_t + (1+g_z)(1+g_n)\tilde{K}_{t+1} - (1-\delta)\tilde{K}_t + Ge^{\hat{g}_t} = Ae^{\hat{a}_t}\tilde{K}_t^{\alpha}L_t^{1-\alpha} & \Rightarrow (4) \\ \tilde{K}^{\text{supply}} = \tilde{K}^{\text{demand}} & \Rightarrow (5) \end{cases}$$

The **system** with 2 endogenous variables $(\hat{k}_{t+1}, \hat{c}_t)$ and we are looking for decisions rule of the form:

$$\hat{k}_{t+1} = \phi_{kk}\hat{k}_t + \phi_{kg}\hat{g}_t + \phi_{ka}\hat{a}_t,$$
$$\hat{c}_t = \phi_{ck}\hat{k}_t + \phi_{cg}\hat{g}_t + \phi_{ca}\hat{a}_t.$$

III) Blanchard and Kahn's Method (1980, The Solution of Linear Difference Models under Rational Expectations)
Define

$$\begin{cases} \mathbf{x}_t \equiv [\mathbf{z}_t, \mathbf{y}_t]', \\ \mathbf{z}_t \equiv [\mathbf{z}_t^1, \mathbf{z}_t^2]', \\ \mathbf{z}_t^1 \equiv \hat{k}_t, & \leftarrow \text{the state variables: endogenous} \\ \mathbf{z}_t^2 \equiv [\hat{g}_t, \hat{a}_t]' & \leftarrow \text{the state variables: exogenous} \\ \mathbf{y}_t \equiv \hat{c}_t. & \leftarrow \text{the control variable: endogenous} \end{cases}$$

$$\mathbf{z}_{t+1}^1 = \mathbb{E}_t\mathbf{z}_{t+1}^1, \qquad \leftarrow \text{predetermined}$$
$$\mathbf{z}_{t+1}^2 = \mathcal{N}\mathbf{z}_t^2 + \boldsymbol{\epsilon}_{t+1}. \quad \leftarrow \text{law of motion}$$

where $\boldsymbol{\epsilon}_t$ is a vector of i.i.d. innovations with mean zero and variance-covariance matrix $\boldsymbol{\Sigma}$.
Thus,

$$\mathcal{A}\mathbf{E}_t\mathbf{x}_{t+1} = \mathcal{B}\mathbf{x}_t \quad \Leftrightarrow \quad [\cdots]\ \mathbf{E}_t \begin{bmatrix} \hat{k}_{t+1} \\ \hat{g}_{t+1} \\ \hat{a}_{t+1} \\ \hat{c}_{t+1} \end{bmatrix} = [\cdots] \begin{bmatrix} \hat{k}_t \\ \hat{g}_t \\ \hat{a}_t \\ \hat{c}_t \end{bmatrix} \quad \Rightarrow \quad \mathbb{E}_t\mathbf{x}_{t+1} = \mathcal{F}\mathbf{x}_t, \quad \text{where } \mathcal{F} \equiv \mathcal{A}^{-1}\mathcal{B}.$$

Solution to the blue **backward looking**/**forward looking** difference equation depends on the eigenvalues of $\mathcal{F}$ (i.e., BK conditions, cf. McCandless, 2008, pp.128- or Pfeifer, 2018, Lecture Notes for Structural Macroeconometrics):

ⓐ If the number of eigenvalues of $\mathcal{F}$ outside/inside the unicircle (explosive/stable) "=" the number of non-predetermined (forward-looking) variables $\rightarrow$ there exisits a unique bounded solution (a locally unique equilibrium);

ⓑ " > " $\rightarrow$ there is no stable solution (no equilibrium);

ⓒ " < " $\rightarrow$ there is an infinity of solutions.

Solving the system by eigenvalue decomposition (Jordan canonical form):

$$\mathcal{F} = \mathcal{D}\mathbf{\Lambda}\mathcal{D}^{-1} \quad \Leftarrow \quad \mathcal{F}\mathcal{D} = \mathbf{\Lambda}\mathcal{D} \quad \Rightarrow \quad (\mathcal{F} - \mathbf{\Lambda}\mathbf{I})\mathcal{D} = \mathbf{0}.$$

$\mathcal{D}$ is a matrix of eigenvectos. The elements on the main diagonal of the matrix $\mathbf{\Lambda}$ are the eigenvalues of $\mathcal{F}$, which are ordered by increasing absolute value. Assume that number of explosive eigenvalues equal to n_y and matrix $\mathcal{F}$ according to stable and explosive eigenvalues:

$$\mathbf{\Lambda} = \begin{bmatrix} \underbrace{\mathbf{\Lambda}_1}_{n_z \times n_z} & \underbrace{\mathbf{0}}_{n_z \times n_y} \\ \underbrace{\mathbf{0}}_{n_y \times n_z} & \underbrace{\mathbf{\Lambda}_2}_{n_y \times n_y} \end{bmatrix}; \qquad \mathcal{F} = \begin{bmatrix} \underbrace{\mathcal{F}_{11}}_{n_z \times n_z} & \underbrace{\mathcal{F}_{12}}_{n_z \times n_y} \\ \underbrace{\mathcal{F}_{21}}_{n_y \times n_z} & \underbrace{\mathcal{F}_{22}}_{n_y \times n_y} \end{bmatrix}; \qquad \mathcal{D}^{-1} = \begin{bmatrix} \underbrace{\mathcal{D}_{11}}_{n_z \times n_z} & \underbrace{\mathcal{D}_{12}}_{n_z \times n_y} \\ \underbrace{\mathcal{D}_{21}}_{n_y \times n_z} & \underbrace{\mathcal{D}_{22}}_{n_y \times n_y} \end{bmatrix}$$

such that all eigenvalues of $\mathbf{\Lambda}_1$ are on or inside the unit circle and all eigenvalues of $\mathbf{\Lambda}_2$ are outside the unit circle.

Then

$$\mathbb{E}_t\mathbf{x}_{t+1} = \mathcal{F}\mathbf{x}_t \quad \Leftrightarrow \quad \mathbb{E}_t\mathcal{D}^{-1}\mathbf{x}_{t+1} = \mathbf{\Lambda}\mathcal{D}^{-1}\mathbf{x}_t \quad \Leftrightarrow \quad \mathbb{E}_t\boldsymbol{\xi}_{t+1} = \mathbf{\Lambda}\boldsymbol{\xi}_t \quad \text{where } \mathcal{D}^{-1}\mathbf{x}_t \equiv \xi_t = \begin{bmatrix} \underbrace{\xi_{1,t}}_{n_z \times 1} \\ \underbrace{\xi_{2,t}}_{n_y \times 1} \end{bmatrix} = \begin{bmatrix} \underbrace{\mathcal{D}_{11}}_{n_z \times n_z} & \underbrace{\mathcal{D}_{12}}_{n_z \times n_y} \\ \underbrace{\mathcal{D}_{21}}_{n_y \times n_z} & \underbrace{\mathcal{D}_{22}}_{n_y \times n_y} \end{bmatrix} \begin{bmatrix} \mathbf{z}_t \\ \mathbf{y}_t \end{bmatrix}$$

$$\mathbb{E}_t\boldsymbol{\xi}_{t+1} = \mathbf{\Lambda}\boldsymbol{\xi}_t \quad \Leftrightarrow \quad \mathbb{E}_t \begin{bmatrix} \boldsymbol{\xi}_{1,t+1} \\ \boldsymbol{\xi}_{2,t+1} \end{bmatrix} = \begin{bmatrix} \mathbf{\Lambda}_1 & 0 \\ 0 & \mathbf{\Lambda}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi}_{1,t} \\ \boldsymbol{\xi}_{2,t} \end{bmatrix} \quad \Leftrightarrow \quad \begin{aligned} \mathbb{E}_t\boldsymbol{\xi}_{1,t+1} &= \mathbf{\Lambda}_1\boldsymbol{\xi}_{1,t} \\ \mathbb{E}_t\boldsymbol{\xi}_{2,t+1} &= \mathbf{\Lambda}_2\boldsymbol{\xi}_{2,t} \end{aligned}$$

As in the univariate case, it is iterated forward,

$$\begin{aligned} \mathbb{E}_t\boldsymbol{\xi}_{2,t+1} &= \mathbf{\Lambda}_2\boldsymbol{\xi}_{2,t}, \quad \text{where } \xi_2 = \mathbf{y} \equiv \hat{c}_t && \leftarrow \text{backward looking} \\ \Rightarrow \quad \boldsymbol{\xi}_{2,t} &= \mathbf{\Lambda}_2^{-1}\mathbb{E}_t\boldsymbol{\xi}_{2,t+1}, && \leftarrow \text{forward looking} \\ &= \mathbf{\Lambda}_2^{-1}\mathbb{E}_t(\mathbf{\Lambda}_2^{-1}\boldsymbol{\xi}_{2,t+2}), \\ &= \mathbf{\Lambda}_2^{-1}\mathbb{E}_t[\mathbf{\Lambda}_2^{-1}(\mathbf{\Lambda}_2^{-1}\boldsymbol{\xi}_{2,t+3})], \\ &\;\vdots \\ &= \lim_{k\to\infty} \mathbf{\Lambda}_2^{-k}\mathbb{E}_t\boldsymbol{\xi}_{2,t+k} \\ &= \mathbf{0}. \qquad \leftarrow |\mathbf{\Lambda}_2^{-1}| < 1 \end{aligned}$$

$$\mathbf{0} = \boldsymbol{\xi}_{2,t} = \mathcal{D}_{21}\mathbf{z}_t + \mathcal{D}_{22}\mathbf{y}_t \quad \Rightarrow \quad \mathbf{y}_t = \mathcal{G}\mathbf{z}_t, \quad \text{where } \mathcal{G} \equiv -\mathcal{D}_{22}^{-1}\mathcal{D}_{21}.$$

We have arrived at a linear policy function for the control variable.

Recall that

$$\begin{aligned} \mathbb{E}_t \begin{bmatrix} \mathbf{z}_{t+1} \\ \mathbf{y}_{t+1} \end{bmatrix} &\equiv \mathbb{E}_t\mathbf{x}_{t+1} = \mathcal{F}\mathbf{x}_t \equiv \begin{bmatrix} \mathcal{F}_{11} & \mathcal{F}_{12} \\ \mathcal{F}_{21} & \mathcal{F}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{z}_t \\ \mathbf{y}_t \end{bmatrix}, \qquad \text{where } \mathcal{F} \equiv \mathcal{A}^{-1}\mathcal{B} \\ \Rightarrow \quad \mathbb{E}_t\mathbf{z}_{t+1} &= \mathcal{F}_{11}\mathbf{z}_t + \mathcal{F}_{12}\mathbf{y}_t, \\ &= \mathcal{F}_{11}\mathbf{z}_t + \mathcal{F}_{12}(-\mathcal{D}_{22}^{-1}\mathcal{D}_{21})\mathbf{z}_t, \\ &= (\mathcal{F}_{11} - \mathcal{F}_{12}\mathcal{D}_{22}^{-1}\mathcal{D}_{21})\mathbf{z}_t, \\ &\equiv \mathcal{H}\mathbf{z}_t, \qquad \mathbf{z}_t \equiv [\hat{k}_t, \hat{g}_t, \hat{a}_t]' \end{aligned}$$

Denote the number of endogenous sate variables with n_{z1} and of the exogenous state variables with n_{z2}.

Recall that

$$\left.\begin{aligned}\mathbf{z}^1_{t+1} &= \mathbb{E}_t\mathbf{z}^1_{t+1}\\ \mathbf{z}^2_{t+1} &= \mathcal{N}\mathbf{z}^2_t + \boldsymbol{\epsilon}_{t+1}\end{aligned}\right\} \quad\Leftrightarrow\quad \mathbf{z}_{t+1} = \mathbb{E}_t\mathbf{z}_{t+1} + \begin{bmatrix}\underbrace{\mathbf{0}}_{n_{z1}\times n_{z2}}\\ \underbrace{\mathbf{I}}_{n_{z1}\times n_{z2}}\end{bmatrix}\boldsymbol{\epsilon}_{t+1} = \mathcal{H}\mathbf{z}_t + \mathcal{I}\boldsymbol{\epsilon}_{t+1}, \quad \text{where } \mathcal{I}\equiv[\underbrace{\mathbf{0}}_{n_{z1}\times n_{z2}}, \underbrace{\mathbf{I}}_{n_{z1}\times n_{z2}}]'$$

Taken together, the solutions discussed above provide a recursive representation of the solution to in state space form

$$\begin{aligned}\mathbf{z}_{t+1} &= \mathcal{H}\mathbf{z}_t + \mathcal{I}\boldsymbol{\epsilon}_{t+1}, && \leftarrow \text{the state/state transition equation}\\ \mathbf{y}_t &= \mathcal{G}\mathbf{z}_t. && \leftarrow \text{the observation/measurement equation}\end{aligned}$$

Given $\mathbf{z}_0$, the state-space representation of the solution can be used to compute the time series which obtains in equilibrium for a given sequence $\{\boldsymbol{\epsilon}_{t+1}\}_{t=0}^{\infty}$ (cf. Pfeifer, 2018, lecture notes for structural macroeconometrics).

IV)

V)

VI)

For more detail, I refer the reader to David and Dave, 2011, Structural Macroeconometrics, 2nd ed., Part II: Model Solution Techniques.

1.8 Steady State and Calibration

$$\begin{aligned}(1+g_z)(1+g_n)\tilde{K}_{t+1} &= \tilde{I}_t + (1-\delta)\tilde{K}_t,\\ \Rightarrow\quad \tilde{I} &= [(1+g_z)(1+g_n)-(1-\delta)]\tilde{K},\\ &= (g_z+g_n+g_zg_n+\delta)\tilde{K},\\ \Rightarrow\quad \delta &= \frac{\tilde{I}}{\tilde{K}} - (g_z+g_n+g_zg_n) = \frac{\tilde{I}/\tilde{Y}}{\tilde{K}/\tilde{Y}} - (g_z+g_n+g_zg_n);\\ \tilde{K}^{\text{supply}} &= \tilde{K}^{\text{demand}},\\ &\Rightarrow \begin{cases}\beta & =\\ \tilde{K} & =\end{cases}\\ L^{\text{supply}} &= L^{\text{demand}},\\ \Rightarrow\quad \varphi &=\\ \tilde{Y}_t &= A_t\tilde{K}_t^{\alpha}L_t^{1-\alpha},\\ \Rightarrow\quad \tilde{Y} &=\\ \tilde{C}_t &= \tilde{Y}_t - \tilde{I}_t - \tilde{G}_t,\\ &= \tilde{Y}_t - [(1+g_z)(1+g_n)\tilde{K}_{t+1} - (1-\delta)\tilde{K}_t] - \tilde{G}_t,\\ \Rightarrow\quad \tilde{C} &= \tilde{Y} - (g_z+g_n+g_zg_n+\delta)K - G.\end{aligned}$$

Pfeifer (2018):

Choose parameter values to make the model consistent with growth observations.

As we want to judge the model's performance to explain business cycles, business cycle observations cannot be used to assign parameter values, i.e., not the same data should be used for calibration and evaluation.

Evaluation of the model on quarterly data (one period in the model corresponds to a quarter).

[3] Source: Pfeifer, 2018, Lecture Notes for Structural Macroeconometrics

Parameter	Value	Target
g_x	0.0055	2.2% Output/Capita Growth
n	0.0027	1.1% Population Growth
A^*	1	Normalization
α	0.33	Capital Share
δ	1.58%	Investment/Output = 0.25
β	0.9926	Capital/Output = 10.4
σ	1	Model-independent evidence
ψ	2.49	$l^* = 0.33$
ρ_z	0.97	Estimate
σ_z	0.0068	Estimate
χ	0.2038	G/Y of 0.2038
ρ_g	0.98	Estimate
σ_g	0.0105	Estimate

Calibration [3]