Macroeconomic Theory
——From the Classical to the New Keynesian

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1 Real Business Cycle Theory

1.1 The representative household

$$\max_{C_t,N_t,B_t} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t), \qquad \beta \in (0,1], \;\; U_{Ct} > 0,\; U_{CCt} < 0, \;\; U_{Nt} < 0,\; U_{NNt} < 0,$$

$$\text{s.t.} \quad P_t C_t + Q_t B_t \leq B_{t-1} + W_t N_t + D_t, \qquad Q_t \equiv \frac{1}{1+i_t} \text{ is the price per bond borught today,}$$

$\Downarrow$ Solving the problem by Extended Kuhn-Tucker conditions or optimal control or dynamic programming

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t [U(C_t, N_t) + \lambda_t (B_{t-1} + W_t N_t + D_t - P_t C_t - Q_t B_t)],$$

$\Downarrow$ FOCs :

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0 \quad \Rightarrow \quad \beta^t (U_{Ct} - \lambda_t P_t) = 0 \quad \Rightarrow \quad \frac{U_{Ct}}{\mathbb{E}_t U_{Ct+1}} = \frac{\lambda_t}{\mathbb{E}_t \lambda_{t+1}} \frac{P_t}{\mathbb{E}_t P_{t+1}},$$

$$\frac{\partial \mathcal{L}}{\partial N_t} = 0 \quad \Rightarrow \quad \beta^t (U_{Nt} + \lambda_t W_t) = 0,$$

$$\frac{\partial \mathcal{L}}{\partial B_t} = 0 \quad \Rightarrow \quad -\lambda_t Q_t \beta^t + \mathbb{E}_t \lambda_{t+1} \beta^{t+1} = 0 \quad \Rightarrow \quad \frac{\beta \mathbb{E}_t \lambda_{t+1}}{\lambda_t} = Q_t = \frac{1}{1+i_t} \quad \Rightarrow \quad \frac{1}{Q_t} = 1 + i_t.$$

$$\lambda_t (B_{t-1} + W_t N_t + D_t - P_t C_t - Q_t B_t) = 0, \qquad \leftarrow \text{ The complementary slackness condition.}$$

$$\left.\begin{aligned} \beta^t (U_{Ct} - \lambda_t P_t) = 0 \\ \beta^t (U_{Nt} + \lambda_t W_t) = 0 \end{aligned}\right\} \Rightarrow -\frac{U_{Nt}}{U_{Ct}} = \frac{W_t}{P_t} \quad \leftarrow \text{ The labor supply equation,}$$

$$\left.\begin{aligned} \frac{U_{Ct}}{\mathbb{E}_t U_{Ct+1}} &= \frac{\lambda_t}{\mathbb{E}_t \lambda_{t+1}} \frac{P_t}{\mathbb{E}_t P_{t+1}} \\ &= \frac{\beta}{Q_t} \frac{P_t}{\mathbb{E}_t P_{t+1}} \end{aligned}\right\} \Rightarrow Q_t = \beta \mathbb{E}_t \left(\frac{U_{Ct+1}}{U_{Ct}} \frac{P_t}{P_{t+1}} \right) \quad \leftarrow \text{ The Euler equation.}$$

$$U(C_t, N_t) = \begin{cases} \frac{C_t^{1-\sigma}-1}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} & \text{for } \sigma \neq 1 \\ \ln C_t - \frac{N_t^{1+\varphi}}{1+\varphi} & \text{for } \sigma = 1 \end{cases} \qquad \rightarrow \qquad \begin{cases} \frac{W_t}{P_t} = C_t^{\sigma} N_t^{\varphi}, \\ Q_t = \beta \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \right]. \end{cases}$$

1.2 The representative firm

$$\min_{N_t} W_t N_t \qquad \Leftrightarrow \qquad \max_{Y_t} \quad P_t Y_t - W_t N_t,$$

$$\text{s.t.} \quad Y_t = A_t \cancel{K_t^{\alpha}} N_t^{1-\alpha}, \qquad \alpha \in [0,1)$$

$$A_t = A_{t-1}^{\rho_a} \epsilon_t^a, \qquad \epsilon_t^a \sim \text{i.i.d.} \mathcal{N}(0, \sigma_a^{\epsilon}).$$

$\Downarrow$FOC:

$$\frac{W_t}{P_t} = (1-\alpha) A_t N_t^{-\alpha} \qquad \leftarrow \text{the labor demand equation.}$$

1.3 Log-linearization

$$\text{AS}\begin{cases}\frac{W_t}{P_t}=(1-\alpha)A_tN_t^{-\alpha} & \Rightarrow \quad \begin{cases} w_t-p_t=\ln(1-\alpha)+a_t-\alpha n_t \\ w-p=\ln(1-\alpha)+a-\alpha n\end{cases} \quad \Rightarrow \quad \hat{w}_t-\hat{p}_t=\hat{a}_t-\alpha\hat{n}_t, \\ \frac{W_t}{P_t}=C_t^{\sigma}N_t^{\varphi} & \Rightarrow \quad \begin{cases} w_t-p_t=\sigma c_t+\varphi n_t \\ w-p=\sigma c+\varphi n\end{cases} \quad \Rightarrow \quad \hat{w}_t-\hat{p}_t=\sigma\hat{c}_t+\varphi\hat{n}_t, \\ Y_t=A_tN_t^{1-\alpha} & \Rightarrow \quad \begin{cases} y_t=a_t+(1-\alpha)n_t \\ y=a+(1-\alpha)n\end{cases} \quad \Rightarrow \quad \hat{y}_t=\hat{a}_t+(1-\alpha)\hat{n}_t, \\ A_t=A_{t-1}^{\rho_a}e^{\epsilon_t^a} & \Rightarrow \quad \hat{a}_t=\rho_a\hat{a}_{t-1}+\epsilon_t^a,\end{cases}$$

$$\text{AD}\begin{cases} Y_t=C_t \quad \Rightarrow \quad \hat{y}_t=\hat{c}_t, \quad \leftarrow \text{The goods market clearing condition} \\ 1=\mathbb{E}_t\beta Q_t^{-1}\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\right] \quad \Rightarrow \quad 1=\mathbb{E}_t\left[e^{\ln\beta-\ln Q_t-\sigma(c_{t+1}-c_t)-(p_{t+1}-p_t)}\right]=? \\ \frac{M_t}{P_t}=\frac{Y_t}{Q_t^{-\eta}} \quad \Rightarrow \quad \hat{m}_t-\hat{p}_t=\hat{y}_t-\eta\hat{i}_t \quad \leftarrow \text{The money demand equation, where } \eta\geq 0 \text{ is the interest rate elasticity.}\end{cases}$$

Notice that the method of taking logs and then substracting the log terms of the steady state equation should not be used on equations that involve expectation terms, even when the equation is multiplicative, because the expectation of a log term is not the same as taking the log of an expectation term.

This is the result of Jensen's inequality, which implies $\ln(\mathbb{E}x \geq \mathbb{E}\ln x)$ for the log function. Only for a linear function $f(x)$ is $f(\mathbb{E}x)=\mathbb{E}f(x)$ (cf. Huang, Lecture Notes).

$$\begin{aligned}
1 &= \mathbb{E}_t\left[e^{\ln\beta-\ln Q_t-\sigma(c_{t+1}-c_t)-(p_{t+1}-p_t)}\right], \\
&= \mathbb{E}_t\left[e^{\ln\beta+i_t-\sigma\Delta c_{t+1}-\pi_{t+1}}\right] \qquad \leftarrow \ln\left(\frac{1}{Q_t}\right)=\ln(1+i_t)\approx i_t, \\
&\Downarrow \text{ a first-order Taylor expansion around steady state that } \ln\beta+i_t-\sigma\Delta c_{t+1}-\pi_t=0 \;\Rightarrow\; -\ln\beta=i_t-\sigma\Delta c_{t+1}-\pi_{t+1}, \\
&= \mathbb{E}_t\{[1+[-\ln\beta-(-\ln\beta)]+(i_t-i)-\sigma(\Delta c_{t+1}-\Delta c)-(\pi_{t+1}-\pi)]\},
\end{aligned}$$

$$\Rightarrow \quad 1=1+\hat{i}_t-\sigma\mathbb{E}_t\Delta\hat{c}_{t+1}-\mathbb{E}_t\hat{\pi}_{t+1},$$

$$\Rightarrow \quad 0=\hat{i}_t-\sigma\mathbb{E}_t\Delta\hat{c}_{t+1}-\mathbb{E}_t\hat{\pi}_{t+1} \quad \Rightarrow \quad \sigma\mathbb{E}_t\Delta\hat{c}_{t+1}=\hat{i}_t-\mathbb{E}_t\hat{\pi}_{t+1} \quad \Rightarrow \quad \mathbb{E}_t\Delta\hat{c}_{t+1}=\frac{1}{\sigma}(\hat{i}_t-\mathbb{E}_t\hat{\pi}_{t+1}),$$

Note that log-linearization means taking the log-deviation around a steady sate value. Assume x denotes the steady sate value of variable x_t. Define the log-deviation of variable x_t from its steady state x as

$$\begin{aligned}
\hat{x}_t &= \ln X_t-\ln X=x_t-x \quad \Rightarrow \quad \ln X_t=\ln X+\hat{x}_t \quad \Rightarrow \quad e^{\ln X_t}=X_t=e^{\ln X+\hat{x}_t}=Xe^{\hat{x}_t}\approx X(1+\hat{x}_t), \\
&= \ln\left(\frac{X_t}{X}\right), \\
&= \ln\left(1+\frac{X_t-X}{X}\right), \\
&\approx \ln 1+\frac{1}{X}(X_t-X), \qquad \leftarrow \text{a first-order Taylor expansion} \\
&= \frac{X_t-X}{X}, \\
&= \frac{X_t}{X}-1,
\end{aligned}$$

$$\Rightarrow \quad \frac{X_t}{X}=1+\hat{x},$$

$$\Rightarrow \quad X_t=X(1+\hat{x}_t).$$

It states that the log deviation of x_t from its steady sate value are approximately equal to the percentage difference between x_t and its steady state value. This approximation holds from small deviations from the steady state, which highlights that log-linearization is a local approximation method.

1.4 The complete model characterized with linearity

$$\begin{cases} \text{AS} & \begin{cases} \hat{w}_t - \hat{p}_t = \hat{a}_t - \alpha \hat{n}_t, \\ \hat{w}_t - \hat{p}_t = \sigma \hat{c}_t + \varphi \hat{n}_t, \\ \hat{y}_t = \hat{a}_t + (1-\alpha)\hat{n}_t, \\ \hat{a}_t = \rho_a \hat{a}_{t-1} + \epsilon_t^a, \end{cases} \\ \text{AD} & \begin{cases} \hat{y}_t = \hat{c}_t, \\ \hat{c}_t = \mathbb{E}_t \hat{c}_{t+1} - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1}), \\ \hat{m}_t - \hat{p}_t = \hat{y}_t - \eta \hat{i}_t, \\ \hat{r}_t = \hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1}, \qquad \leftarrow \text{ The Fisherian equation.} \end{cases} \end{cases}$$

1.5 Determination of equilibrium values of real variables

Labor market clearing $N^s = N^d = N$, or $\hat{n}^s = \hat{n}^d = n$, so that

$$\begin{aligned}
\hat{a}_t - \alpha \hat{n}_t &= \sigma \hat{c}_t + \varphi \hat{n}_t, \\
&= \sigma \hat{y}_t + \varphi \hat{n}_t, \\
&= \sigma[\hat{a}_t + (1-\alpha)\hat{n}_t] + \varphi \hat{n}_t, \\
\Rightarrow \quad \hat{n}_t &= \frac{1-\sigma}{\sigma(1-\alpha)+\alpha+\varphi}\hat{a}_t; \\
\Rightarrow \quad \hat{y}_t &= \hat{a}_t + (1-\alpha)\frac{1-\sigma}{\sigma(1-\alpha)+\alpha+\varphi}\hat{a}_t, \\
&= \frac{\sigma(1-\alpha)+\alpha+\varphi+(1-\sigma)(1-\alpha)}{\sigma(1-\alpha)+\alpha+\varphi}\hat{a}_t, \\
&= \frac{1+\varphi}{\sigma(1-\alpha)+\alpha+\varphi}\hat{a}_t; \\
&= \hat{c}_t; \\
\hat{r}_t &= \sigma \mathbb{E}_t \Delta \hat{c}_{t+1}, \qquad \leftarrow \text{ the Euler equation} \\
&= \sigma \mathbb{E}_t \Delta \hat{y}_{t+1}, \\
&= \sigma \frac{1+\varphi}{\sigma(1-\alpha)+\alpha+\varphi}\mathbb{E}_t(\hat{a}_{t+1} - \hat{a}_t), \\
&= \sigma \frac{1+\varphi}{\sigma(1-\alpha)+\alpha+\varphi}\mathbb{E}_t(\rho_a \hat{a}_t + \epsilon_{t+1}^a - \hat{a}_t), \\
&= \sigma \frac{1+\varphi}{\sigma(1-\alpha)+\alpha+\varphi}\mathbb{E}_t[(\rho_a - 1)\hat{a}_t + \epsilon_{t+1}^a], \\
&= \sigma(\rho_a - 1)\frac{1+\varphi}{\sigma(1-\alpha)+\alpha+\varphi}\hat{a}_t, \qquad \leftarrow \mathbb{E}_t \epsilon_{t+1}^a = 0; \\
\hat{\omega}_t &\equiv \hat{w}_t - \hat{p}_t, \\
&= \hat{a}_t - \alpha \hat{n}_t, \\
&= \hat{a}_t - \alpha \frac{1-\sigma}{\sigma(1-\alpha)+\alpha+\varphi}\hat{a}_t, \\
&= \frac{\sigma(1-\alpha)+\alpha+\varphi-\alpha(1-\sigma)}{\sigma(1-\alpha)+\alpha+\varphi}\hat{a}_t, \\
&= \frac{\sigma+\varphi}{\sigma(1-\alpha)+\alpha+\varphi}\hat{a}_t.
\end{aligned}$$

Gali (2015, slides/book):

Neutrality: real variables determined independently of monetary policy;

Optimal Monetary Policy: undetermined;

In contrast with real variables, the equilibrium values of nominal variables cannot be determined independently of monetary policy.

1.6 Monetary policy and price level determination

Example i: An exogenous path for the nominal interest rate:

$$
\begin{aligned}
i_t &= i + \upsilon_t \quad \Rightarrow \quad \hat{i}_t = \upsilon_t,\\
\upsilon_t &= \rho_\upsilon \upsilon_{t-1} + \epsilon_t^\upsilon, \qquad \epsilon_t^\upsilon \sim \text{i.i.d.}\mathcal{N}(0, \sigma_\upsilon^\epsilon).\\
&\Downarrow\\
\mathbb{E}_t\hat{\pi}_{t+1} &= \hat{i}_t - \hat{r}_t,\\
&= \upsilon_t - \hat{r}_t,\\
&\Uparrow\\
\Rightarrow \quad \hat{\pi}_t &= \upsilon_{t-1} - \hat{r}_{t-1} + \xi_t, \quad \text{with } \mathbb{E}_t\xi_{t+1} = 0 \text{ for all } t,\\
&\Downarrow\\
&\begin{cases} \hat{p}_t = \hat{p}_{t-1} + \upsilon_{t-1} - \hat{r}_{t-1} + \xi_t,\\ \hat{w}_t = \hat{\omega}_t + \hat{p}_t. \end{cases}
\end{aligned}
$$

where ξ_t is a **sunspot shock**, possibly unrelated to economic fundamentals. Note that expected inflation is determined since it can be written as a function of exogenous variables. But actual inflation is not. An equilibrium in which such nonfundamental factors may cause fluctuations in one or more variables is referred to as an **indeterminate equilibrium**. The example above shows how an exogenous nominal interest rate leads to nominal variables indeterminacy.

Example ii: A simple interest rate rule:

$$
\begin{aligned}
i_t &= i + \phi_\pi \hat{\pi}_t + \upsilon_t \quad \Rightarrow \quad \hat{i}_t = \phi_\pi\hat{\pi}_t + \upsilon_t, \qquad \phi_\pi > 0,\\
\upsilon_t &= \rho_\upsilon \upsilon_{t-1} + \epsilon_t^\upsilon, \qquad \rho_\upsilon \in [0,1], \quad \epsilon_t^\upsilon \sim \text{i.i.d.}\mathcal{N}(0, \sigma_\upsilon^\epsilon).\\
&\Downarrow
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}_t\hat{\pi}_{t+1} &= \hat{i}_t - \hat{r}_t,\\
&= \phi_\pi\hat{\pi}_t + \upsilon_t - \hat{r}_t,\\
&\downarrow
\end{aligned}
$$

$$
\begin{cases}
\hat{\pi}_t = \dfrac{1}{\phi_\pi}(\mathbb{E}_t\hat{\pi}_{t+1} + \hat{r}_t - \upsilon_t), \qquad \text{if } \phi_\pi > 1,\\
\Rightarrow \quad \hat{\pi}_t = \displaystyle\sum_{k=0}^{\infty}\phi_\pi^{-(k+1)}\mathbb{E}_t(\hat{r}_{t+k} - \upsilon_{t+k}), \qquad \leftarrow \text{solve forward to get only one nonexplosive solution}\\
\quad = \displaystyle\sum_{k=0}^{\infty}\phi_\pi^{-(k+1)}\mathbb{E}_t\left[\sigma(\rho_a - 1)\frac{1+\varphi}{\sigma(1-\alpha)+\alpha+\varphi}\hat{a}_{t+k} - \upsilon_{t+k}\right],\\
\quad = \sigma(\rho_a - 1)\dfrac{1+\varphi}{\sigma(1-\alpha)+\alpha+\varphi}\displaystyle\sum_{k=0}^{\infty}\phi_\pi^{-(k+1)}\mathbb{E}_t\hat{a}_{t+k} - \sum_{k=0}^{\infty}\phi_\pi^{-(k+1)}\mathbb{E}_t\upsilon_{t+k}, \quad \hat{a}_t = \rho_a\hat{a}_{t-1} + \epsilon_t^a,\ \upsilon_t = \rho_\upsilon\upsilon_{t-1} + \epsilon_t^\upsilon,\\
\quad = \sigma(\rho_a - 1)\dfrac{1+\varphi}{\sigma(1-\alpha)+\alpha+\varphi}\mathbb{E}_t(\phi_\pi^{-1}\hat{a}_t + \phi_\pi^{-2}\hat{a}_{t+1} + \phi_\pi^{-3}\hat{a}_{t+2} + \cdots) - \mathbb{E}_t(\phi_\pi^{-1}\upsilon_t + \phi_\pi^{-2}\upsilon_{t+1} + \phi_\pi^{-3}\upsilon_{t+2} + \cdots),\\
\quad = \sigma(\rho_a - 1)\dfrac{1+\varphi}{\sigma(1-\alpha)+\alpha+\varphi}\mathbb{E}_t[\phi_\pi^{-1}\hat{a}_t + \phi_\pi^{-2}(\rho_a\hat{a}_t + \epsilon_{t+1}^a) + \cdots] - \mathbb{E}_t[\phi_\pi^{-1}\upsilon_t + \phi_\pi^{-2}(\rho_\upsilon\upsilon_t + \epsilon_{t+1}^\upsilon) + \cdots],\\
\quad = \sigma(\rho_a - 1)\dfrac{1+\varphi}{\sigma(1-\alpha)+\alpha+\varphi}(\phi_\pi^{-1}\hat{a}_t + \phi_\pi^{-2}\rho_a\hat{a}_t + \phi_\pi^{-3}\rho_a^2\hat{a}_t + \cdots) - (\phi_\pi^{-1}\upsilon_t + \phi_\pi^{-2}\rho_\upsilon\upsilon_t + \phi_\pi^{-3}\rho_\upsilon^2\upsilon_t + \cdots),\\
\quad = \sigma(\rho_a - 1)\dfrac{1+\varphi}{\sigma(1-\alpha)+\alpha+\varphi}(\phi_\pi^{-1} + \phi_\pi^{-2}\rho_a + \phi_\pi^{-3}\rho_a^2 + \cdots)\hat{a}_t - (\phi_\pi^{-1} + \phi_\pi^{-2}\rho_\upsilon + \phi_\pi^{-3}\rho_\upsilon^2 + \cdots)\upsilon_t,\\
\quad = \sigma(\rho_a - 1)\dfrac{1+\varphi}{\sigma(1-\alpha)+\alpha+\varphi}\dfrac{\phi_\pi^{-1}}{1-\phi_\pi^{-1}\rho_a}\hat{a}_t - \dfrac{\phi_\pi^{-1}}{1-\phi_\pi^{-1}\rho_\upsilon}\upsilon_t,\\
\quad = \sigma(\rho_a - 1)\dfrac{1+\varphi}{\sigma(1-\alpha)+\alpha+\varphi}\dfrac{1}{\phi_\pi-\rho_a}\hat{a}_t - \dfrac{1}{\phi_\pi-\rho_\upsilon}\upsilon_t,\\
\quad \begin{cases} \Rightarrow \quad \hat{p}_t = \hat{p}_{t-1} + \hat{\pi}_t, \qquad \Leftarrow \ \hat{\pi}_t = \hat{p}_t - \hat{p}_{t-1},\\ \Rightarrow \quad \hat{w}_t = \hat{\omega}_t + \hat{p}_t. \end{cases};\\
\mathbb{E}_t\hat{\pi}_{t+1} = \phi_\pi\hat{\pi}_t + \upsilon_t - \hat{r}_t, \qquad \text{if } 0 < \phi_\pi < 1 \ \text{(nominal indeterminacy)},\\
\Rightarrow \quad \hat{\pi}_t = \phi_\pi\hat{\pi}_{t-1} + \upsilon_{t-1} - \hat{r}_{t-1} + \xi_t, \quad \text{with } \mathbb{E}_t\xi_{t+1} = 0 \text{ for all } t,\\
\quad \begin{cases} \Rightarrow \quad \hat{p}_t = \hat{p}_{t-1} + \hat{\pi}_t,\\ \Rightarrow \quad \hat{w}_t = \hat{\omega}_t + \hat{p}_t. \end{cases}
\end{cases}
$$

Responses to monetary policy shock ($\phi_\pi > 1$, i.e., **the "Taylor principle" requirement**)

$$\begin{aligned}
i_t &= i + \phi_\pi \hat{\pi}_t + \upsilon_t, \\
&= i + \phi_\pi \left[\sigma(\rho_a - 1) \frac{1+\varphi}{\sigma(1-\alpha)+\alpha+\varphi} \frac{1}{\phi_\pi - \rho_a} \hat{a}_t - \frac{1}{\phi_\pi - \rho_\upsilon} \upsilon_t \right] + \upsilon_t, \\
&= i + \phi_\pi \sigma(\rho_a - 1) \frac{1+\varphi}{\sigma(1-\alpha)+\alpha+\varphi} \frac{1}{\phi_\pi - \rho_a} \hat{a}_t - \phi_\pi \frac{1}{\phi_\pi - \rho_\upsilon} \upsilon_t + \upsilon_t, \\
&= i + \phi_\pi \sigma(\rho_a - 1) \frac{1+\varphi}{\sigma(1-\alpha)+\alpha+\varphi} \frac{1}{\phi_\pi - \rho_a} \hat{a}_t + \left(1 - \frac{\phi_\pi}{\phi_\pi - \rho_\upsilon} \right) \upsilon_t, \\
&= i + \phi_\pi \sigma(\rho_a - 1) \frac{1+\varphi}{\sigma(1-\alpha)+\alpha+\varphi} \frac{1}{\phi_\pi - \rho_a} \hat{a}_t - \frac{\rho_\upsilon}{\phi_\pi - \rho_\upsilon} \upsilon_t; \\
\hat{m}_t &= \hat{p}_t + \hat{c}_t - \eta \hat{i}_t, \\
&= \hat{p}_{t-1} + \hat{\pi}_t + \hat{y}_t - \eta \hat{i}_t, \\
&= \hat{p}_{t-1} + \left[\sigma(\rho_a - 1) \frac{1+\varphi}{\sigma(1-\alpha)+\alpha+\varphi} \frac{1}{\phi_\pi - \rho_a} \hat{a}_t - \frac{1}{\phi_\pi - \rho_\upsilon} \upsilon_t \right] \\
&\quad + \frac{1+\varphi}{\sigma(1-\alpha)+\alpha+\varphi} \hat{a}_t - \eta \left[\phi_\pi \sigma(\rho_a - 1) \frac{1+\varphi}{\sigma(1-\alpha)+\alpha+\varphi} \frac{1}{\phi_\pi - \rho_a} \hat{a}_t - \frac{\rho_\upsilon}{\phi_\pi - \rho_\upsilon} \upsilon_t \right], \\
&= \cdots + \frac{\eta \rho_\upsilon - 1}{\phi_\pi - \rho_\upsilon} \upsilon_t.
\end{aligned}$$

For simplicity, let $\rho_\upsilon = 0$ and $\upsilon_t = \epsilon_t^\upsilon$ which means the process of monetary shock is i.i.d.

Then we arrive at

$$\begin{cases}
\frac{\partial \hat{y}_t}{\partial \epsilon_t^\upsilon} = 0, \\
\frac{\partial i_t}{\partial \epsilon_t^\upsilon} = -\frac{\rho_\upsilon}{\phi_\pi - \rho_\upsilon} = 0, \\
\frac{\partial \hat{\pi}_t}{\partial \epsilon_t^\upsilon} = -\frac{1}{\phi_\pi - \rho_\upsilon} = -\frac{1}{\phi_\pi} < 0, \\
\frac{\partial \hat{m}_t}{\partial \epsilon_t^\upsilon} = \frac{\eta \rho_\upsilon - 1}{\phi_\pi - \rho_\upsilon} = -\frac{1}{\phi_\pi} < 0.
\end{cases}$$

Turn to the case in which $0 < \rho_\upsilon < 1$ which means the shock is persistent but damping.

$$\begin{cases}
\frac{\partial \hat{y}_t}{\partial \epsilon_t^\upsilon} = 0, \\
\frac{\partial i_t}{\partial \epsilon_t^\upsilon} = -\frac{\rho_\upsilon}{\phi_\pi - \rho_\upsilon} < 0, \quad \to \text{ liquidity effect} \\
\frac{\partial \hat{\pi}_t}{\partial \epsilon_t^\upsilon} = -\frac{1}{\phi_\pi - \rho_\upsilon} < 0, \\
\frac{\partial \hat{m}_t}{\partial \epsilon_t^\upsilon} = \frac{\eta \rho_\upsilon - 1}{\phi_\pi - \rho_\upsilon} \lesseqgtr 0.
\end{cases}$$

And the case $\rho_\upsilon = 1$? (The shock will last forever)

$$\begin{cases}
\frac{\partial \hat{y}_t}{\partial \epsilon_t^\upsilon} = 0, \\
\frac{\partial i_t}{\partial \epsilon_t^\upsilon} = -\frac{\rho_\upsilon}{\phi_\pi - \rho_\upsilon} = -\frac{1}{\phi_\pi - 1} < 0, \\
\frac{\partial \hat{\pi}_t}{\partial \epsilon_t^\upsilon} = -\frac{1}{\phi_\pi - \rho_\upsilon} = -\frac{1}{\phi_\pi - 1} < 0, \\
\frac{\partial \hat{m}_t}{\partial \epsilon_t^\upsilon} = \frac{\eta \rho_\upsilon - 1}{\phi_\pi - \rho_\upsilon} = \frac{\eta - 1}{\phi_\pi - 1} \lesseqgtr 0, \quad \text{recall that } \eta \geq 0.
\end{cases}$$

The analysis above assumed that monetary policy can be described by an interest rule. In the above two cases, money supply does not play an independent role in determining the equilibrium and just adjusts endogenously.

Example iii: Another simple interest rate rule,

$$
\begin{aligned}
\hat{m}_t - \hat{p}_t = \hat{y}_t - \eta\hat{i}_t \quad &\overset{\eta=0}{\Longrightarrow} \quad \hat{m} = \hat{p}_t + \hat{y}_t,\\
\mathbb{E}_t\Delta\hat{c}_{t+1} = \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t\hat{\pi}_{t+1}), \quad \hat{y}_t = \hat{c}_t \quad &\overset{\sigma=1}{\Longrightarrow} \quad \hat{i}_t = \mathbb{E}_t(\Delta\hat{c}_{t+1} + \hat{\pi}_{t+1}) = \mathbb{E}_t\Delta\hat{m}_{t+1}.\\
i_t = i + \phi\Delta\hat{m} - \sigma_\upsilon^\epsilon\upsilon_t \quad &\Rightarrow \quad \hat{i}_t = \phi_m\Delta\hat{m} - \sigma_\upsilon^\epsilon\upsilon_t,\ \phi > 1,\ \upsilon_t \sim \mathcal{N}(0,1)\ \ \upsilon_t = \rho_\upsilon\upsilon_{t-1} + \epsilon_t^\upsilon,\ \rho_\upsilon = 0,\ \epsilon_t^\upsilon \sim \mathcal{N}(0,\sigma_\upsilon^\epsilon).\\
&\Downarrow\\
\phi_m\Delta\hat{m} - \sigma_\upsilon^\epsilon\upsilon_t = \mathbb{E}_t\Delta\hat{m}_{t+1} \quad &\Rightarrow \quad \Delta\hat{m}_t = \phi_m^{-1}\mathbb{E}_t\Delta\hat{m}_{t+1} + \frac{\sigma_\upsilon^\epsilon}{\phi_m}\upsilon_t,\\
&\Downarrow \text{ iterate forward and note that } \lim_{k\to\infty}\phi_m^{-k}\mathbb{E}_t\Delta\hat{m}_{t+k} = 0\\
\Delta\hat{m}_t = \frac{\sigma_\upsilon^\epsilon}{\phi_m}\upsilon_t \quad &\Leftrightarrow \quad \underbrace{\hat{m}_t = \hat{m}_{t-1} + \frac{\sigma_\upsilon^\epsilon}{\phi_m}\upsilon_t}_{\text{an equilibrium outcome}} \quad \leftrightarrow \quad \Delta\hat{m}_t = \rho_m\Delta\hat{m}_{t-1} + \epsilon_t \quad \Leftrightarrow \quad \text{ARMA}(2,2).
\end{aligned}
$$

A difference equation with rational expectation:

$$
\begin{aligned}
&\vdots\\
x_{t+2} &= y^{-1}x_{t+3} + cz_{t+2},\\
x_{t+1} &= y^{-1}x_{t+2} + cz_{t+1},\\
x_t &= y^{-1}x_{t+1} + cz_t,\\
&= y^{-1}(y^{-1}x_{t+2} + cz_{t+1}) + cz_t\\
&= y^{-2}x_{t+2} + y^{-1}z_{t+1} + cz_t,\\
&= y^{-2}(y^{-1}x_{t+3} + cz_{t+2}) + y^{-1}z_{t+1} + cz_t,\\
&= y^{-3}x_{t+3} + y^{-2}cz_{t+2} + y^{-1}z_{t+1} + cz_t,\\
&\vdots\\
&= \underbrace{y^{-k}x_{t+k}}_{0 \text{ when } k\to\infty} + \underbrace{y^{-(k-1)}cz_{t+k-1} + \cdots + y^{-1}z_{t+1}}_{\mathbb{E}_t z_{t+k-1}=0} + cz_t.
\end{aligned}
$$

Example iv: An exogenous path for the money supply:

$$\hat{m}_t - \hat{p}_t = \hat{y}_t - \eta\hat{i}_t, \qquad \eta > 0, \qquad \eta = 0 \;\Rightarrow\; \hat{m}_t - \hat{p}_t = \hat{y}_t,$$

$$\begin{aligned}
\Rightarrow \quad \hat{p}_t &= \hat{m}_t - \hat{y}_t + \eta\hat{i}_t,\\
&= \hat{m}_t - \hat{y}_t + \eta(\mathbb{E}_t\hat{\pi}_{t+1} + \hat{r}_t),\\
&= \hat{m}_t - \hat{y}_t + \eta(\mathbb{E}_t\hat{p}_{t+1} - \hat{p}_t + \hat{r}_t),\\
\Rightarrow \quad \hat{p}_t &= \frac{\eta}{1+\eta}\mathbb{E}_t\hat{p}_{t+1} + \frac{1}{1+\eta}\hat{m}_t + \frac{1}{1+\eta}(\eta\hat{r}_t - \hat{y}_t),\\
&= \frac{\eta}{1+\eta}\mathbb{E}_t\hat{p}_{t+1} + \frac{1}{1+\eta}\hat{m}_t + u_t, \qquad \text{where } u_t \equiv \frac{1}{1+\eta}(\eta\hat{r}_t - \hat{y}_t) \text{ evolves independently of } \{\hat{m}_t\},\\
&= \frac{1}{1+\eta}\sum_{k=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^k \mathbb{E}_t\hat{m}_{t+k} + \sum_{k=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^k \mathbb{E}_t u_{t+k}.\\
\hat{p}_t &= \frac{1}{1+\eta}\sum_{k=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^k \mathbb{E}_t\hat{m}_{t+k} + \bar{u}_t, \quad \text{where } \bar{u}_t \equiv \sum_{k=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^k \mathbb{E}_t u_{t+k} \text{ is independent of monetary policy.}\\
\hat{p}_t &= \frac{\eta}{1+\eta}\mathbb{E}_t\hat{p}_{t+1} + \frac{1}{1+\eta}\hat{m}_t + u_t,\\
\Rightarrow \quad \hat{p}_t - \hat{m}_t &= \frac{\eta}{1+\eta}\mathbb{E}_t\hat{p}_{t+1} + \frac{1}{1+\eta}\hat{m}_t + u_t + \hat{m}_{t+1} - \hat{m}_{t+1} - \hat{m}_t,\\
&= \frac{\eta}{1+\eta}\mathbb{E}_t\hat{p}_{t+1} + \frac{1}{1+\eta}\hat{m}_t + u_t + \left(\frac{\eta}{1+\eta} + \frac{1}{1+\eta}\right)(\hat{m}_{t+1} - \hat{m}_{t+1} - \hat{m}_t),\\
&= \frac{\eta}{1+\eta}\mathbb{E}_t\hat{p}_{t+1} + \frac{1}{1+\eta}\hat{m}_t + u_t + \frac{\eta}{1+\eta}(\hat{m}_{t+1} - \hat{m}_{t+1} - \hat{m}_t) + \frac{1}{1+\eta}(\hat{m}_{t+1} - \hat{m}_{t+1} - m_t),\\
&= \frac{\eta}{1+\eta}\mathbb{E}_t(\hat{p}_{t+1} - \hat{m}_{t+1}) + \frac{\eta}{1+\eta}\mathbb{E}(\hat{m}_{t+1} - \hat{m}_t) + \frac{1}{1+\eta}\mathbb{E}(\hat{m}_{t+1} - \hat{m}_{t+1}) + \frac{1}{1+\eta}(\hat{m}_t - \hat{m}_t) + u_t,\\
&= \frac{\eta}{1+\eta}\mathbb{E}_t(\hat{p}_{t+1} - \hat{m}_{t+1}) + \frac{\eta}{1+\eta}\mathbb{E}(\hat{m}_{t+1} - \hat{m}_t) + u_t,\\
\Rightarrow \quad \hat{p}_t - \hat{m}_t &= \frac{\eta}{1+\eta}\mathbb{E}_t(\hat{p}_{t+1} - \hat{m}_{t+1}) + \frac{\eta}{1+\eta}\mathbb{E}(\hat{m}_{t+1} - \hat{m}_t) + u_t,\\
\Rightarrow \quad \tilde{p}_t &= \frac{\eta}{1+\eta}\mathbb{E}_t\tilde{p}_{t+1} + \frac{\eta}{1+\eta}\mathbb{E}\Delta\hat{m}_{t+1} + u_t, \qquad \text{where } \tilde{p}_t \equiv \hat{p}_t - \hat{m}_t,\\
&= \sum_{k=1}^{\infty}\left(\frac{\eta}{1+\eta}\right)^k \mathbb{E}_t\Delta\hat{m}_{t+k} + \bar{u}_t, \quad \leftarrow \text{ can be solved forward to yield}\\
\Rightarrow \quad p_t &= \hat{m}_t + \sum_{k=1}^{\infty}\left(\frac{\eta}{1+\eta}\right)^k \mathbb{E}_t\Delta\hat{m}_{t+k} + \bar{u}_t;\\
\hat{m}_t - \hat{p}_t &= \hat{y}_t - \eta\hat{i}_t,\\
\Rightarrow \quad \hat{i}_t &= \frac{1}{\eta}[\hat{y}_t - (\hat{m}_t - \hat{p}_t)],\\
&= \frac{1}{\eta}(\hat{p}_t - \hat{m}_t) + \frac{1}{\eta}\hat{y}_t,\\
&= \frac{1}{\eta}\left[\sum_{k=1}^{\infty}\left(\frac{\eta}{1+\eta}\right)^k \mathbb{E}_t\Delta\hat{m}_{t+k} + \bar{u}_t\right] + \frac{1}{\eta}\hat{y}_t,\\
&= \frac{1}{\eta}\sum_{k=1}^{\infty}\left(\frac{\eta}{1+\eta}\right)^k \mathbb{E}_t\Delta\hat{m}_{t+k} + \frac{1}{\eta}(\hat{y}_t + \bar{u}_t),\\
&= \frac{1}{\eta}\sum_{k=1}^{\infty}\left(\frac{\eta}{1+\eta}\right)^k \mathbb{E}_t\Delta\hat{m}_{t+k} + \tilde{u}_t, \quad \text{where } \tilde{u}_t \equiv \frac{1}{\eta}(\hat{y}_t + \bar{u}_t) \text{ is independent of monetary policy.}
\end{aligned}$$

1.7 Optimal Monetary Policy

Given that household's utility is a function of C and N only——two real variables that are invariant to the way monetary policy is conducted——it follows that no policy rule is better than any other (cf, Gali, 2015, ch.2.4.4).

1.8 Money in the Utility Function (MIC)

We will discuss it in DNK model and solve it by dynamic programming.